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Vector multivariate subdivision schemes: Comparison of spectral methods for their regularity analysis

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ABSTRACT

We study vector multivariate subdivision schemes with dilation $2I$ satisfying sum rules of order $k+1$ and multiplicity m. It is well known that the magnitude of the associated joint spectral radius or, alternatively, the magnitude of the associated restricted spectral radius characterizes the W_p^k-regularity, $k \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$, of such a scheme. This characterization alone does not necessarily indicate any intrinsic connection between the two radii. In this paper, we unify the two approaches based on the concepts of the joint spectral radius and the restricted spectral radius and show that these two numbers are equal. Therefore, the only difference between these approaches is that they offer different numerical schemes for estimating the regularity of subdivision. We show how to obtain the restricted spectral radius estimates using the techniques of linear programming and convex minimization. We illustrate our results with several examples.

1. Introduction

Subdivision schemes are computational means for generating finer and finer meshes in $\mathbb{R}^s$, usually in dimension $s=2,3$. At each step of the subdivision recursion, the topology of the finer mesh is inherited from the coarser mesh and the coordinates of the finer vertices, stored in a vector sequence $\mathbf{c}^{(r+1)}$, are computed by local averages of the coarser ones

$$\mathbf{c}^{(r+1)} = S_{\boldsymbol{A}}\mathbf{c}^{(r)} = \sum_{\beta\in\mathbb{Z}^s} A(\cdot-2\beta)c^{(r)}(\beta), \quad r \geqslant 0.$$

Subdivision rules are described by the finite matrix sequence $\boldsymbol{A} = (A(\alpha))_{\alpha\in\mathbb{Z}^s}$, the so-called subdivision mask. The locality and algorithmic simplicity of the subdivision recursion ensure that it is fast, efficient, and easy to implement. These features explain the increasing popularity of subdivision in computer graphics, computer aided geometrical design and multiresolution analysis for wavelet and frame constructions, see [2,7,12–14] for details.

In the multivariate case, it still remains a challenging task to characterize the regularity of subdivision schemes in such a way that it also yields computationally accessible method for checking their regularity. How one approaches this task depends both on the underlying topology of the mesh and on the local averaging rules stored in $\boldsymbol{A}$. We study the shift-invariant setting, i.e. the mesh is isomorphic to $\mathbb{Z}^s$, and the definition of $S_{\boldsymbol{A}}$ is independent of either the position of the newly inserted vertex or of the level r of the subdivision recursion. We assume that the dimension m of the joint 1-eigenspace of the $n\times n$ real matrices

$$A_\varepsilon = \sum_{\alpha\in\mathbb{Z}^s} A(\varepsilon-2\alpha), \quad \varepsilon\in\{0,1\}^s,$$

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satisfies $1 \leqslant m \leqslant n$. The regularity analysis in such setting is done in terms of the subdivision mask $\boldsymbol{A}$ and usually employs either the concept of the joint spectral radius (JSR) of a finite set of matrices derived from the mask, see [6,21,23], or restricted spectral properties (RSR), sometimes referred to as contractivity, of so-called difference subdivision schemes, which are also obtained from the mask [2,4,14]. It has been believed until recently that the JSR and the RSR approaches are intrinsically different. The main contribution of this paper is that it unifies these approaches and shows that they characterize the W_p^k-regularity, $k \in \mathbb{N}$, of subdivision in terms of the same quantity. Furthermore, we demonstrate that the JSR and the RSR approaches differ only by the numerical schemes they provide for the estimation of this quantity. We also show that the problem of obtaining such RSR estimates is equivalent to solving a problem of linear programming, when $p = \infty$, or to a problem of convex minimization, when $1 \leqslant p < \infty$. Our results extend the ones sketched in the proceedings paper [5] for the L_p-convergence.

The paper is organized as follows: In Section 2 we introduce some basic notation and facts about vector multivariate subdivision schemes with dilation $2I$. In the case $k = 0$, the assumption on $1 \leqslant m \leqslant n$ above is important for the study in [5] and states that certain constant polynomial sequences are the eigensequences of the subdivision operator $S_{\boldsymbol{A}}$. This property of $S_{\boldsymbol{A}}$ ensures the existence of the so-called first difference schemes. In the case $k \geqslant 1$, the study of the structure of the polynomial eigenspaces of $S_{\boldsymbol{A}}$ of degree k is crucial for the existence of the higher order difference schemes and the subsequent comparison of the JSR and the RSR. One of the main results of Section 3 states that the existence of the difference subdivision schemes is ensured, if the mask of the original scheme satisfies certain sum rules of order $k+1$ and multiplicity m. We also recall a practical method [20] for checking this property of the mask and show how to transform the mask appropriately, if such sum rules are not satisfied. In Section 4 we introduce the concept of the restricted (k, p)-spectral radius and explain why this quantity allows us to characterize the smoothness of subdivision, see also [2,4]. We also provide the estimates for the (k, p)-restricted norm of the iterated subdivision operator. These estimates improve the ones given in [5]. They also allow us to show that the JSR and the RSR are equal and yield a numerical scheme for the estimation of the RSR. In Section 5, we illustrate our results with several examples. In the case $p = \infty$, these show that our optimization based scheme for the RSR offers a practical and efficient alternative to the numerical method for estimating of the JSR. A more detailed comparison of the above mentioned numerical methods is under further investigation.

2. Notation and background

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An element $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is a multi-index whose length is given by $|\mu| := \mu_1 + \cdots + \mu_s$, $\mu! := \mu_1! \ldots \mu_s!$ and

$$\binom{\mu}{\nu} = \frac{\mu!}{\nu!(\mu-\nu)!}, \quad \mu, \nu \in \mathbb{N}_0^s,\ \nu_j \leqslant \mu_j,\ j = 1, \ldots, s.$$

For $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$ and $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ define $\alpha^\mu := \alpha_1^{\mu_1} \ldots \alpha_s^{\mu_s}$. We denote by ϵ_ℓ, $\ell = 1, \ldots, s$, and by e_j, $j = 1, \ldots, n$, the standard unit vectors of $\mathbb{R}^s$ and $\mathbb{R}^n$, respectively. The operator D^μ stands for the mixed partial derivative $D_{\epsilon_1}^{\mu_1} \ldots D_{\epsilon_s}^{\mu_s}$.

By $(W_p^k(\mathbb{R}^s))^n$ denote the linear space of all vector-valued functions whose components are the elements of

$$W_p^k(\mathbb{R}^s) = \{ f \in L_p(\mathbb{R}^s)\colon\ D^\mu f \in L_p(\mathbb{R}^s),\ \mu \in \mathbb{N}_0^s,\ |\mu| \leqslant k \}, \quad 1 \leqslant p \leqslant \infty.$$

Let $\ell^{n\times k}(\mathbb{Z}^s)$ denote the linear space of all sequences of $n \times k$ real matrices indexed by $\mathbb{Z}^s$. In addition, let $\ell_p^{n\times k}(\mathbb{Z}^s)$ denote the Banach space of sequences of $n \times k$ real matrices indexed by $\mathbb{Z}^s$ with finite p-norm defined as

$$\|\boldsymbol{C}\|_p := \begin{cases} (\sum_{\alpha\in\mathbb{Z}^s} |C(\alpha)|_p^p)^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{\alpha\in\mathbb{Z}^s} |C(\alpha)|_\infty, & p = \infty, \end{cases} \tag{1}$$

where $|C(\alpha)|_p$ is the p-operator norm if $k > 1$ and the p-vector norm if $k = 1$. For notational simplicity, we write $\ell_p^n(\mathbb{Z}^s)$ for $\ell_p^{n\times 1}(\mathbb{Z}^s)$ and denote vector sequences by lowercase bold letters. Moreover, let $\ell_0^{n\times k}(\mathbb{Z}^s) \subset \ell_p^{n\times k}(\mathbb{Z}^s)$ be the space of finitely supported matrix valued sequences. Specific examples of such scalar and matrix sequences are the sequences $\boldsymbol{\delta} \in \ell_0(\mathbb{Z}^s)$ and $\boldsymbol{\delta} I_n \in \ell_0^{n\times n}(\mathbb{Z}^s)$

$$\delta(\alpha) := \begin{cases} 1, & \alpha = 0, \\ 0, & \alpha \in \mathbb{Z}^s \setminus \{0\} \end{cases} \quad \text{and} \quad \delta I_n(\alpha) := \begin{cases} I_n, & \alpha = 0, \\ 0, & \alpha \in \mathbb{Z}^s \setminus \{0\}. \end{cases}$$

For a finite set $K \subset \mathbb{Z}^s$ we denote by $\ell^{n\times k}(K) \subset \ell_0^{n\times k}(\mathbb{Z}^s)$ the linear space of all sequences supported in K.

The *subdivision operator* $S_{\boldsymbol{A}} : \ell^n(\mathbb{Z}^s) \to \ell^n(\mathbb{Z}^s)$ associated with the *mask* $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ is defined by

$$S_{\boldsymbol{A}}\boldsymbol{c}(\alpha) = \sum_{\beta\in\mathbb{Z}^s} A(\alpha - 2\beta)c(\beta), \quad \alpha \in \mathbb{Z}^s.$$

It follows from the finite support of $\boldsymbol{A}$ that the linear operator $S_{\boldsymbol{A}}$ is a bounded operator from $\ell_p^n(\mathbb{Z}^s)$ to itself. We assume that the mask is shifted so that $\mathrm{supp}(\boldsymbol{A}) \subset [0, N]^s$ and $A(0) \neq 0$.

The *subdivision scheme* then corresponds to a repeated application of $S_{\boldsymbol{A}}$ to a starting vector sequence $\boldsymbol{c} \in \ell^n(\mathbb{Z}^s)$ yielding

$$\boldsymbol{c}^{(0)} := \boldsymbol{c}, \qquad \boldsymbol{c}^{(r+1)} := S_{\boldsymbol{A}}\boldsymbol{c}^{(r)}, \quad r \geqslant 0. \tag{2}$$

The scheme in (2) can be also expressed by means of the *iterated mask* $\boldsymbol{A}^{(r)} \in \ell_0^{n\times n}(\mathbb{Z}^s)$

$$\boldsymbol{c}^{(r+1)} = S_{\boldsymbol{A}}^r \boldsymbol{c}(\alpha) = \sum_{\beta\in\mathbb{Z}^s} A^{(r)}\big(\alpha - 2^r\beta\big)c(\beta), \quad \alpha \in \mathbb{Z}^s,$$

with

$$A^{(r)}(\alpha) := \sum_{\beta\in\mathbb{Z}^s} A^{(r-1)}(\beta)A(\alpha - 2\beta), \quad r \geqslant 1, \qquad \boldsymbol{A}^{(0)} := \boldsymbol{\delta} I_n. \tag{3}$$

The iteration in (2), when applied to a starting matrix sequence $\boldsymbol{C} \in \ell^{n\times k}(\mathbb{Z}^s)$, can be understood as the application of $S_{\boldsymbol{A}}$ to the columns of $\boldsymbol{C}^{(r)}$ separately and storing the results accordingly in $\boldsymbol{C}^{(r+1)}$.

We define the $n \times n$ real matrices

$$A_\varepsilon := \sum_{\alpha\in\mathbb{Z}^s} A(\varepsilon - 2\alpha), \quad \varepsilon \in \{0,1\}^s \cap \mathbb{Z}^s,$$

and their joint 1-eigenspace $\mathcal{E}_{\boldsymbol{A}} := \{v \in \mathbb{R}^n\colon\ A_\varepsilon v = v,\ \varepsilon \in \{0,1\}^s\}$. It has been shown in [11] that it is necessary for the convergence of $S_{\boldsymbol{A}}$ that $\mathcal{E}_{\boldsymbol{A}}$ is non-trivial. Equivalently, this states that it is necessary for the convergence of $S_{\boldsymbol{A}}$ that one step of the subdivision recursion reproduces constant sequences $\boldsymbol{v}$ with $v(\alpha) = v$, $v \in \mathcal{E}_{\boldsymbol{A}}$, $\alpha \in \mathbb{Z}^s$. The assumption that $\mathcal{E}_{\boldsymbol{A}} \neq \{0\}$ is also known as the sum rules of order 1.

A useful tool for studying subdivision schemes is the Laurent polynomial formalism. For a finite matrix sequence $\boldsymbol{A} \in \ell_0^{n\times k}(\mathbb{Z}^s)$ we define the associated *symbol* as the *Laurent polynomial*

$$A^*(z) := 2^{-s} \sum_{\alpha\in\mathbb{Z}^s} A(\alpha)z^\alpha, \quad z \in \big(\mathbb{C}\setminus\{0\}\big)^s,\ z^\alpha = z_1^{\alpha_1}\cdot\ldots\cdot z_s^{\alpha_s}.$$

For our regularity analysis we make use of the properties of the so-called difference schemes. Such schemes are defined using the notion of the backward difference operator, the discrete analog of a derivative. Its definition depends on $m := \dim\mathcal{E}_{\boldsymbol{A}}$. For $\boldsymbol{C} \in \ell^{n\times d}(\mathbb{Z}^s)$ and $\boldsymbol{D} \in \ell^{d\times n}(\mathbb{Z}^s)$ we define the j-th column of $\nabla_\ell \boldsymbol{C}$ as

$$[\nabla_\ell \boldsymbol{C}]_{\cdot,j} := \begin{bmatrix} \boldsymbol{C}_{1,j} - \boldsymbol{C}_{1,j}(\cdot - \epsilon_\ell) \\ \vdots \\ \boldsymbol{C}_{m,j} - \boldsymbol{C}_{m,j}(\cdot - \epsilon_\ell) \\ \boldsymbol{C}_{m+1,j} \\ \vdots \\ \boldsymbol{C}_{n,j} \end{bmatrix}, \quad 1 \leqslant \ell \leqslant s,\ 1 \leqslant j \leqslant d,$$

and $\nabla_\ell \boldsymbol{D} := (\nabla_\ell \boldsymbol{D}^T)^T$, respectively. The *backward difference operator* $\nabla : \ell^{n\times d}(\mathbb{Z}^s) \to \ell^{ns\times d}(\mathbb{Z}^s)$ is then defined by

$$\nabla := \begin{bmatrix} \nabla_1 \\ \vdots \\ \nabla_s \end{bmatrix} \quad \text{with } \nabla^*(z) = \begin{bmatrix} (1-z_1)I_m & 0 \\ 0 & I_{n-m} \\ \vdots & \\ (1-z_s)I_m & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

We also define the *iterated backward difference operator*. To this purpose we consider the *Kronecker product* of operators. (We recall that for two matrices $A \otimes B$ is the block matrix with the block representation $[A_{ij}B]_{ij}$.) For $k \in \mathbb{N}_0$, define the *k-th order backward difference* operator

$$\nabla^k : \ell^{n\times d}\big(\mathbb{Z}^s\big) \to \ell^{ns^k\times d}\big(\mathbb{Z}^s\big), \quad d \in \mathbb{N},$$

by

$$\nabla^k := \bigotimes_{\ell=1}^{k} \nabla = \left(\bigotimes_{\ell=1}^{k-1} \nabla\right) \otimes \nabla, \quad \bigotimes_{\ell=1}^{1} \nabla := \nabla,\ \nabla^0 := \mathrm{id}.$$

The matrix Laurent polynomial associated with ∇^k is given by

$$\left(\nabla^k\right)^*(z) = \prod_{j=1}^{k} I_{s^{j-1}} \otimes \nabla^*(z).$$

In the scalar multivariate case, the operator $\nabla^k : \ell(\mathbb{Z}^s) \to \ell^{s^k}(\mathbb{Z}^s)$ is the column vector with the entries being all possible k-th backward differences. The ordering of these differences is determined by the structure of the Kronecker products in the above Laurent polynomials for ∇^k. To fix this ordering, we denote by $\mathcal{M}_k \in \mathbb{N}_0^{s^k \times s}$ with $\mathcal{M}_0 := 0$ the matrix whose j-th row stores $\alpha \in \mathbb{N}_0^s$ with its ℓ-entry, α_ℓ, being equal to the number of times the operator ∇_ℓ appears in the rows $n(j-1)+1$ to $n(j-1)+n$ of ∇^k.

To comply with the notation in [6] we define for $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ the difference operator

$$\widetilde{\nabla}^\mu : \ell^{n\times d}\left(\mathbb{Z}^s\right) \to \ell^{n\times d}\left(\mathbb{Z}^s\right), \quad \widetilde{\nabla}^\mu := \prod_{\ell=1}^{s} \nabla_\ell^{\mu_\ell}$$

with $\nabla_\ell^{\mu_\ell}$ given by $\nabla_\ell^{\mu_\ell} = \nabla_\ell \cdot (\nabla_\ell^{\mu_\ell - 1})$, $\mu_\ell \in \mathbb{N}$, $\nabla_\ell^0 := \mathrm{id}$.

We say that the subdivision scheme $S_{\boldsymbol{A}}$ is *W_p^k-convergent*, $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$, if for any starting sequence $\boldsymbol{c} \in \ell_p^n(\mathbb{Z}^s)$ there exists a vector-valued function $f_{\boldsymbol{c}} \in (W_p^k(\mathbb{R}^s))^n$ such that for any test function $g \in W_\infty^k(\mathbb{R}^s)$

$$\lim_{r\to\infty} \max_{|\mu|\leqslant k} \left\| D^\mu f_{\boldsymbol{c}} - g I_n * \left(2^{|\mu| r} \widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{c}\right)\left(2^r \cdot\right) \right\|_p = 0,$$

see [11] for the definition of test functions. We use the Fourier domain formulation of sum rules with the Fourier transform of a matrix sequence $\boldsymbol{C} \in \ell^{n\times k}(\mathbb{Z}^s)$ defined by

$$\widehat{C}(\xi) = 2^{-s} \sum_{\beta\in\mathbb{Z}^s} C(\alpha)\left(e^{-i\xi}\right)^\alpha, \quad \xi \in \mathbb{R}^s, \ \left(e^{-i\xi}\right)^\alpha = \left(e^{-i\xi_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(e^{-i\xi_s}\right)^{\alpha_s}.$$

We say, analogously to [15], that $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ satisfies the *sum rules of order $k+1$ and multiplicity m*, if there exist sequences $\boldsymbol{y}_j \in \ell_0^n(\mathbb{Z}^s)$, $j = 1, \ldots, m$, with linearly independent vectors $\hat{y}_j(0)$, $j = 1, \ldots, m$, such that

$$D^\mu\left[\widehat{A}(\cdot)\hat{y}_j(2\cdot)\right](2\pi\beta) = \delta(2\beta) D^\mu \hat{y}_j(0), \tag{4}$$

for $\mu \in \mathbb{N}_0^s$, $|\mu| \leqslant k$, and $\beta \in (\frac{1}{2}\mathbb{Z}^s \setminus \mathbb{Z}^s) \cup \{0\}$.

In the case $m = 1$, the sum rules have been widely used for studying subdivision schemes and their properties, see [2, 15,19,22] and references therein. The equivalence between the so-called time domain [25] and Fourier domain formulations of sum rules is established in e.g. [23, pp. 12–13]. The results [6, Theorem 3.2] and [15, Proposition 2.5 Part 6)] combined together show that sum rules of higher order are necessary for regularity of vector multivariate subdivision. These results make use of the so-called *Eigenvalue Condition*, see e.g. [6, p. 19], and are proven for general isotropic integer dilation matrices. To our knowledge, there are no results in the literature that show that sum rules are necessary for W_p^k-regularity of multivariate subdivision schemes whose symbol does not satisfy *Eigenvalue Condition*, e.g., the scheme in [9], or in the case $1 < m \leqslant n$.

3. Difference schemes

In this section, we address the crucial issue of the existence of the difference schemes $S_{\boldsymbol{B}_k}$ satisfying

$$\nabla^k S_{\boldsymbol{A}} = S_{\boldsymbol{B}_k} \nabla^k, \quad 1 \leqslant k < N, \tag{5}$$

and characterize their existence in terms of the eigenspaces of $S_{\boldsymbol{A}}$, see Theorem 3.8. If the limits of the appropriately scaled difference schemes $S_{\boldsymbol{B}_k}$ derived from $S_{\boldsymbol{A}}$ exist, then they yield the partial derivatives of the limits of $S_{\boldsymbol{A}}$. Thus, it is natural to study the properties of $S_{\boldsymbol{B}_k}$ to determine the W_p^{k-1}-regularity of $S_{\boldsymbol{A}}$. We start by deriving an equivalent formulation of the sum rules (4).

Proposition 3.1. *The mask $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ satisfies sum rules (4) of order $k+1$ and multiplicity m if and only if the system*

$$\left(\widehat{A}(\pi\varepsilon) - \delta(\varepsilon)\frac{1}{2^{|\mu|}} I_n\right) w_{j,\mu} = -\sum_{\substack{0\leqslant\nu\leqslant\mu \\ \nu\neq 0}} \binom{\mu}{\nu} (2i)^{-|\nu|} D^\nu \widehat{A}(\pi\varepsilon) w_{j,\mu-\nu}, \tag{6}$$

is solvable for each $\varepsilon \in \{0,1\}^s$, $j = 1, \ldots, m$, $\mu \in \mathbb{N}_0^s$, $|\mu| \leqslant k$, and the solutions $w_{j,0}$, $j = 1, \ldots, m$, are linearly independent.

Proof. $\Rightarrow$: Let $j=1,\dots,m$ and $\mu \in \mathbb{N}_0^s$, $|\mu| \leqslant k$. Using Leibniz differentiation formula we get that sum rules are equivalent to

$$\sum_{\nu\in\mathbb{N}_0^s,\, \nu\leqslant\mu} \binom{\mu}{\nu} D^\nu \widehat{A}(2\pi\beta)\bigl[D^{\mu-\nu}\hat{y}_j(2\cdot)\bigr](2\pi\beta) = \delta(2\beta) D^\mu \hat{y}_j(0)$$

with $\beta \in (\frac{1}{2}\mathbb{Z}^s \setminus \mathbb{Z}^s) \cup \{0\}$. Note that

$$\begin{aligned}\bigl[D^{\mu-\nu}\hat{y}_j(2\cdot)\bigr](2\pi\beta) &= 2^{-s}\Biggl[D^{\mu-\nu}\sum_{\gamma\in\mathbb{Z}^s} y_j(\gamma)\bigl(e^{-i2\cdot}\bigr)^\gamma\Biggr](2\pi\beta) \\ &= 2^{-s+|\mu|-|\nu|}\sum_{\gamma\in\mathbb{Z}^s}(-i\gamma)^{|\mu|-|\nu|} y_j(\gamma)\bigl(e^{-i4\pi\beta}\bigr)^\gamma = 2^{|\mu|-|\nu|}\bigl[D^{\mu-\nu}\hat{y}_j\bigr](0)\end{aligned}$$

for all $\beta \in (\frac{1}{2}\mathbb{Z}^s \setminus \mathbb{Z}^s) \cup \{0\}$. This and $\frac{1}{2}\mathbb{Z}^s \setminus \mathbb{Z}^s = \bigcup_{\alpha\in\mathbb{Z}^s}(\alpha + (\{0,1/2\}^s)\setminus\{0\})$ imply that it suffices to consider $\beta \in \{0,1/2\}^s$ and (4) is equivalent to

$$\bigl(\widehat{A}(\pi\varepsilon) - \delta(\varepsilon)2^{-|\mu|}I\bigr)D^\mu\hat{y}_j(0) = -\sum_{\substack{0\leqslant\nu\leqslant\mu\\ \nu\neq 0}}\binom{\mu}{\nu}2^{-|\nu|}\, D^\nu\widehat{A}(\pi\varepsilon)D^{\mu-\nu}\hat{y}_j(0)$$

with $\varepsilon \in \{0,1\}^s$, see also [25, (3.4)]. This implies the solvability of (6) for $w_{j,\mu} = i^{-|\mu|}\, D^\mu\hat{y}_j(0)$ for all $\varepsilon\in\{0,1\}^s$, $j=1,\dots,m$, and $|\mu|\leqslant k$.

$\Leftarrow$: Assume that (6) is solvable for $\varepsilon\in\{0,1\}^s$, $j=1,\dots,m$ and $|\mu|\leqslant k$. To determine the sequences $\boldsymbol{y}_j \in \ell_0^n(\mathbb{Z}^s)$, solve the systems $D^\mu\hat{y}_j(0) = i^{|\mu|}\, w_{j,\mu}$. Equivalently, to determine the ℓ-th components of the sequences $\boldsymbol{y}_j\in\ell_0^n(\mathbb{Z}^s)$ solve

$$\bigl[i^{-|\tilde\mu|}D^{\tilde\mu}e^{-i\mu\cdot}(0)\bigr]_{\substack{\tilde\mu,\mu\in\mathbb{N}_0^s\\ |\mu|\leqslant k,\,|\tilde\mu|\leqslant k}}\cdot\bigl[\bigl(y_j(\mu)\bigr)_\ell\bigr]_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|\leqslant k}} = \bigl[(w_{j,\tilde\mu})_\ell\bigr]_{\substack{\tilde\mu\in\mathbb{N}_0^s\\ |\tilde\mu|\leqslant k}} \tag{7}$$

with

$$\bigl[D^{\tilde\mu}e^{-i\mu\cdot}(0)\bigr]_{\substack{\tilde\mu,\mu\in\mathbb{N}_0^s\\ |\mu|\leqslant k,\,|\tilde\mu|\leqslant k}} = \bigl[(-i)^{|\tilde\mu|}\mu^{\tilde\mu}\bigr]_{\substack{\tilde\mu,\mu\in\mathbb{N}_0^s\\ |\mu|\leqslant k,\,|\tilde\mu|\leqslant k}}.$$

Note that the multivariate Vandermonde matrix $[(-1)^{|\tilde\mu|}\mu^{\tilde\mu}]_{\substack{\tilde\mu,\mu\in\mathbb{N}_0^s\\ |\mu|\leqslant k,|\tilde\mu|\leqslant k}}$ is invertible. The system of equations in (7) determines $y_j(\beta)$ for $\beta\in[0,k]^s$. The rest of the elements of $\boldsymbol{y}_j$ we set to zero. $\square$

Remark 3.2. By Proposition 3.1 and by [20, Section 3.4], the sum rules (4) are equivalent to the existence of polynomial eigensequences $\boldsymbol{x}_{j,\mu}$ of $S_{\boldsymbol{A}}$, i.e. $S_{\boldsymbol{A}}\boldsymbol{x}_{j,\mu} = 2^{-|\mu|}\boldsymbol{x}_{j,\mu}$, $j=1,\dots,m$, $\mu\in\mathbb{N}_0^s$, $|\mu|\leqslant k$, of the form

$$x_{j,\mu}(\alpha) := \sum_{\substack{\nu\in\mathbb{N}_0^s\\ 0\leqslant\nu\leqslant\mu}}\binom{\mu}{\nu} w_{j,\nu}\alpha^{\mu-\nu},\quad \alpha\in\mathbb{Z}^s \tag{8}$$

with $w_{j,\nu} = i^{-|\nu|}\, D^\nu\hat{y}_j(0)$, $|\nu|\leqslant k$, see also [15,25].

The proof of Proposition 3.3 yields an algorithm for transforming the mask $\boldsymbol{A}$ so that the eigensequences $\tilde{\boldsymbol{x}}_{j,\mu}$ of the transformed subdivision operator $S_{\widetilde{\boldsymbol{A}}}$ satisfy (8) with

$$w_{j,0} = e_j,\qquad w_{j,\nu}\in\operatorname{span}\{e_j\colon\ j=1,\dots,m\},\ \nu\in\mathbb{N}_0^s,\ |\nu|\leqslant k. \tag{9}$$

The property (9) is equivalent to the fact that the corresponding $\tilde{\boldsymbol{y}}_j$ satisfy (11). The proof of Proposition 3.3 uses the idea of [15, Proposition 2.4] proved for $m=1$.

Proposition 3.3. *Let $\boldsymbol{y}_j\in\ell_0^n(\mathbb{Z}^s)$ be such that the vectors $\hat{y}_j(0)$, $j=1,\dots,m$, are linearly independent. Then there exist matrix sequences $\boldsymbol{T}, \boldsymbol{T}_{inv}\in\ell_0^{n\times n}(\mathbb{Z}^s)$ such that*

$$\widehat{T}(\xi)\cdot\widehat{T}_{inv}(\xi) = \widehat{T}_{inv}(\xi)\cdot\widehat{T}(\xi) = I_n,\quad \xi\in\mathbb{R}^s, \tag{10}$$

and the components of $\hat{\tilde{y}}_j(\xi) = \widehat{T}(\xi)\hat{y}_j(\xi)$ satisfy for $j=1,\dots,m$ and $\ell=1,\dots,n$

$$\bigl(\hat{\tilde{y}}_j(0)\bigr)_\ell = \delta(\ell-j),\qquad \bigl(D^\mu\hat{\tilde{y}}_j(0)\bigr)_\ell = 0,\quad \ell\neq j,\ \mu\in\mathbb{N}_0^s,\ |\mu|\leqslant k. \tag{11}$$

Proof. Choose an $n \times n$ permutation matrix P such that the upper $m \times m$ block of the $n \times m$ matrix $P \cdot [\hat{y}_1(0) \ \dots \ \hat{y}_m(0)]$ is invertible.

Set $\widehat{T}(\xi) = \widehat{T}_{inv}(\xi) = P$.

Repeat for $j_1 = 1, \dots, m$:

Set $\hat{\tilde{y}}_j(\xi) = \widehat{T}(\xi)\hat{y}_j(\xi)$, $j = 1, \dots, m$, $\xi \in \mathbb{R}^s$. Due to linear independence of $\hat{y}_j(0)$, $j = 1, \dots, m$, and the invertibility of $\widehat{T}(0)$ there exists an index $j_2 \in \{1, \dots, m\}$ such that the j_1-th component $(\hat{\tilde{y}}_{j_2}(0))_{j_1} \neq 0$.

Determine a sequence $\boldsymbol{c} \in \ell_0^{n-1}(\mathbb{Z}^s)$ such that its components $c_{j_3} \in \ell_0(\mathbb{Z}^s)$ satisfy

$$D^\mu\big[-\hat{c}_{j_3}(\hat{\tilde{y}}_{j_2})_{j_1} + (\hat{\tilde{y}}_{j_2})_{j_3}\big](0) = 0, \quad j_3 = 1, \dots, n, \ j_3 \neq j_1.$$

To do so, solve for $j_3 = 1, \dots, n$, $j_3 \neq j_1$, the linear systems

$$D^\mu \hat{c}_{j_3}(0) = \left[D^\mu \frac{(\hat{\tilde{y}}_{j_2})_{j_3}}{(\hat{\tilde{y}}_{j_2})_{j_1}}\right](0), \quad |\mu| \leqslant k. \tag{12}$$

The matrix on the left-hand side of (12) is the Vandermonde matrix in (7) and is, thus, invertible. Note that the support of the sequence $\boldsymbol{c}$ is the same as the support of $\tilde{\boldsymbol{y}}_{j_2}$. Modify

$$\widehat{T}(\xi) = \begin{bmatrix} I_{j_1-1} & [-\hat{c}_{j_3}(\xi)]_{1\leqslant j_3\leqslant j_1-1} & 0_{j_1-1\times n-j_1} \\ 0 & 1 & 0 \\ 0_{n-j_1\times j_1-1} & [-\hat{c}_{j_3}(\xi)]_{j_1+1\leqslant j_3\leqslant n} & I_{n-j_1} \end{bmatrix} \cdot \widehat{T}(\xi)$$

and

$$\widehat{T}_{inv}(\xi) = \widehat{T}_{inv}(\xi) \cdot \begin{bmatrix} I_{j_1-1} & [\hat{c}_{j_3}(\xi)]_{1\leqslant j_3\leqslant j_1-1} & 0_{j_1-1\times n-j_1} \\ 0 & 1 & 0 \\ 0_{n-j_1\times j_1-1} & [\hat{c}_{j_3}(\xi)]_{j_1+1\leqslant j_3\leqslant n} & I_{n-j_1} \end{bmatrix}.$$

Then (10) is satisfied.

End repeat

Sort the functions $\widehat{T}(\xi)\hat{y}_j(\xi)$, $j = 1, \dots, m$, and the columns of $\widehat{T}$ and $\widehat{T}_{inv}$ accordingly, so that $[\widehat{T}(\xi)\ \hat{y}_j(\xi)]_j \neq 0$. Afterwards, to ensure (11) we modify $\widehat{T}(\xi)$ once more setting

$$\widehat{T}(\xi) = \begin{bmatrix} \operatorname{diag}([\frac{1}{(\widehat{T}(0)\hat{y}_j(0))_j}]_{1\leqslant j\leqslant m}) & 0 \\ 0 & I_{n-m} \end{bmatrix} \widehat{T}(\xi)$$

and

$$\widehat{T}_{inv}(\xi) = \widehat{T}_{inv}(\xi) \begin{bmatrix} \operatorname{diag}([(\widehat{T}(0)\hat{y}_j(0))_j]_{1\leqslant j\leqslant m}) & 0 \\ 0 & I_{n-m} \end{bmatrix}.$$

Now all $\widehat{T}(\xi)\ \hat{y}_j(\xi)$, $j = 1, \dots, m$, satisfy (11) and (10) still holds. $\square$

Example 5.1 shows that one cannot expect that the sequences $\boldsymbol{y}_j$ always satisfy (11). Therefore, using Proposition 3.3 with the sequences $\boldsymbol{y}_j$ satisfying the sum rules (4) we define a transformed mask $\widetilde{\boldsymbol{A}}$ by

$$\widehat{\widetilde{A}}(\xi) = \widehat{T}(\xi) \cdot \widehat{A}(\xi) \cdot \widehat{T}_{inv}(2\xi), \quad \xi \in \mathbb{R}^s. \tag{13}$$

Note that $\widetilde{\boldsymbol{A}} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ due to $\operatorname{supp}(\boldsymbol{y}_j) \subset [0,k]^s$ and

$$\operatorname{supp}(\boldsymbol{T}) = \operatorname{supp}(\boldsymbol{T}_{inv}) \subset [0, mk]^s,$$

which imply $\operatorname{supp}(\widetilde{\boldsymbol{A}}) \subset [0, N+3mk]^s$. The properties of $\widetilde{\boldsymbol{A}}$ are crucial for our subsequent analysis. The results of Lemma 3.4 and Proposition 3.5 show that the regularity and other properties of $S_{\boldsymbol{A}}$ are characterized by those of $S_{\widetilde{\boldsymbol{A}}}$. The proofs of these results follows directly from the properties of $\widehat{T}$, see [3].

Lemma 3.4. *Let $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$. The following statements are equivalent*:

(i) *$\boldsymbol{A}$ satisfies sum rules of order $k+1$ and multiplicity m with $\hat{y}_j(\xi)$ for $j = 1, \dots, m$.*
(ii) *$\widetilde{\boldsymbol{A}}$ in (13) satisfies sum rules of order $k+1$ and multiplicity m with $\hat{\tilde{y}}_j(\xi) = \widehat{T}(\xi)\hat{y}_j(\xi)$, $j = 1, \dots, m$.*

(iii) *For $0 < |\mu| \leqslant k$ and $\beta \in (\frac{1}{2}\mathbb{Z}^s \setminus \mathbb{Z}^s) \cup \{0\}$, the mask satisfies*

$$\widehat{\widetilde{A}}(2\pi\beta) = \begin{bmatrix} \delta(2\beta)I_m & * \\ 0_{n-m,m} & * \end{bmatrix} \quad \text{and} \quad D^{\mu}\widehat{\widetilde{A}}(2\pi\beta) = \begin{bmatrix} \delta(2\beta)E & * \\ 0_{n-m,m} & * \end{bmatrix} \tag{14}$$

for some $m \times m$ diagonal matrix E.

Note that if the mask satisfies (14), then it suffices to solve (6) only for $\varepsilon = 0$. The other equations in (6) are then of the form $0 = 0$, as a consequence of the special structure of the matrices $D^{\nu}\widehat{A}(\pi\varepsilon)$, $\varepsilon \in \{0,1\}^s \setminus \{0\}$.

Proposition 3.5. *Assume that $\boldsymbol{A}$ satisfies the sum rules (4) of order $k+1$ and multiplicity m with $\boldsymbol{y}_j \in \ell_0^n(\mathbb{Z}^s)$, $1 \leqslant j \leqslant m$, and $\widetilde{\boldsymbol{A}}$ satisfies (13). Then*

(i) $\mathcal{E}_{\widetilde{\boldsymbol{A}}} = \operatorname{span}\{e_j\colon 1 \leqslant j \leqslant m\}$ *if and only if* $\mathcal{E}_{\boldsymbol{A}} = \operatorname{span}\{\hat{y}_j(0)\colon 1 \leqslant j \leqslant m\}$.
(ii) *The eigenvalues of $\widehat{A}(0)$ and $\widehat{\widetilde{A}}(0)$ and their multiplicities are the same.*
(iii) $S_{\widetilde{\boldsymbol{A}}}\boldsymbol{x}_{j,\mu} = 2^{-|\mu|}\boldsymbol{x}_{j,\mu}$, $|\mu| \leqslant k$, $j = 1,\dots,m$, *with $\boldsymbol{x}_{j,\mu}$ as in (8) if and only if* $S_{\boldsymbol{A}}\boldsymbol{x}_{j,\mu} = 2^{-|\mu|}\boldsymbol{x}_{j,\mu}$, $|\mu| \leqslant k$, $j = 1,\dots,m$, *with $\boldsymbol{x}_{j,\mu}$ defined by (9).*
(iv) $S_{\boldsymbol{A}}$ *is W_p^k-convergent if and only if $S_{\widetilde{\boldsymbol{A}}}$ is W_p^k-convergent.*

To state the sufficient and necessary conditions for the existence of the difference schemes, see Theorem 3.8, we study certain eigenspaces of $S_{\boldsymbol{A}}$, or, equivalently, the algebraic properties of the mask symbol. Recall that we assume that the mask is shifted appropriately so that the symbol $A^*(z)$ is a polynomial. To state the algebraic properties of this polynomial we need to define the polynomial ideal

$$\mathcal{I} = \langle 1 - z^2 \rangle = \langle 1 - z_\ell^2,\ \ell = 1,\dots,s \rangle$$

and the quotient ideal

$$\mathcal{J} = \mathcal{I}\colon \quad \langle 1 - z_\ell,\ \ell = 1,\dots,s \rangle,$$

i.e., $f \in \mathcal{J}$ if and only if $f \cdot (1 - z_\ell) \in \mathcal{I}$ for all $\ell = 1,\dots,s$, see [10]. The following auxiliary result is crucial for the proof of Theorem 3.8 and gives an insight into the structure of the relevant eigenspaces of the difference masks.

Theorem 3.6. *Let $1 \leqslant m \leqslant n$, $k \geqslant 1$ and $\boldsymbol{B}_{k-1} \in \ell_0^{ns^{k-1} \times ns^{k-1}}(\mathbb{Z}^s)$, $\boldsymbol{B}_0 := \boldsymbol{A}$. The following statements are equivalent:*

(i) *There exists a mask $\boldsymbol{B}_k \in \ell_0^{ns^k \times ns^k}(\mathbb{Z}^s)$ satisfying*

$$\left[I_{s^{k-1}} \otimes \nabla^*(z)\right] B_{k-1}^*(z) = B_k^*(z)\left[I_{s^{k-1}} \otimes \nabla^*\left(z^2\right)\right] \tag{15}$$

and the polynomials $(B_{k-1}^*(z))_{ij}$ belong to the ideal $\mathcal{I}$ for $i = 1,\dots,ns^{k-1}$ and $j = 1 + n\ell,\dots,m + n\ell$ with $\ell = 0,\dots,s^{k-1} - 1$ and $i \neq j$.

(ii) *The joint eigenspace of some non-zero eigenvalues of the matrices*

$$B_{k-1,\varepsilon} := \sum_{\beta \in \mathbb{Z}^s} B_{k-1}(\varepsilon + 2\beta), \quad \varepsilon \in \{0,1\}^s,$$

has the dimension $m \cdot s^{k-1}$ and is spanned by the columns of

$$\{I_{s^{k-1}} \otimes e_j\colon\ j = 1,\dots,m\}. \tag{16}$$

Proof. The fact that the joint eigenspace of the matrices $B_{k-1,\varepsilon}$ is spanned by column vectors in (16) is equivalent to the fact that the matrices $B_{k-1,\varepsilon} \in \mathbb{R}^{s^{k-1}n \times s^{k-1}n}$ have the following structure

$$B_{k-1,\varepsilon} = [B_{ij}]_{1 \leqslant i,j \leqslant s^{k-1}} \tag{17}$$

with

$$B_{ij} = \begin{cases} \left[\begin{smallmatrix} E & *_{m \times n-m} \\ 0_{n-m \times m} & *_{n-m \times n-m} \end{smallmatrix}\right], & i = j, \\ \left[\begin{smallmatrix} 0_{m \times m} & *_{m \times n-m} \\ 0_{n-m \times m} & *_{n-m \times n-m} \end{smallmatrix}\right], & \text{otherwise}, \end{cases}$$

where the matrix E is diagonal with the corresponding non-zero eigenvalues of $B_{k-1,\varepsilon}$ on the diagonal and the matrices given by $*$ vary for different k, ε and depending on the position in $B_{k-1,\varepsilon}$. By definition of $\widehat{B}_{k-1}$ we get

$$\widehat{B}_{k-1}(2\pi\beta) = 2^{-s} \sum_{\varepsilon\in\{0,1\}^s} B_{k-1,\varepsilon}\left(e^{-i2\pi\beta}\right)^{\varepsilon}, \quad \beta \in \{0,1/2\}^s.$$

Note that the first m columns of all B_{ij}, $1 \leqslant i, j \leqslant s^{k-1}$, are independent of ε. Thus, the identities

$$\sum_{\beta\in\{0,1/2\}^s} \left(e^{-i2\pi\beta}\right)^{\varepsilon} = \delta(\varepsilon)2^s, \quad \varepsilon \in \{0,1\}^s,$$
$$\sum_{\varepsilon\in\{0,1\}^s} \left(e^{-i2\pi\beta}\right)^{\varepsilon} = \delta(2\beta)\,2^s, \quad \beta \in \{0,1/2\}^s,$$

yield that (17) is satisfied if and only if the Fourier transforms

$$\widehat{B}_{k-1}(\xi) = \left[\widehat{B}_{ij}(\xi)\right]_{1\leqslant i,j\leqslant s^{k-1}}$$

satisfy

$$\widehat{B}_{ij}(2\pi\beta) = \begin{cases} \begin{bmatrix} \delta(2\beta)\cdot E & *_{m\times n-m} \\ 0_{n-m\times m} & *_{n-m\times n-m} \end{bmatrix}, & i=j, \\ \begin{bmatrix} 0_{m\times m} & *_{m\times n-m} \\ 0_{n-m\times m} & *_{n-m\times n-m} \end{bmatrix}, & \text{otherwise}, \end{cases} \tag{18}$$

for $\beta \in \{0,1/2\}^s$. Set $z = e^{-i\xi}$ and shift the mask $\boldsymbol{B}_{k-1}$ so that the corresponding symbol $B^*_{k-1}(z)$ is a polynomial. Then, by [26, Theorem 2, Proposition 3], (18) holds if and only if the appropriately shifted $n \times n$ blocks $B^*_{ij}(z)$ of $B^*_{k-1}(z)$ satisfy the algebraic property

$$B^*_{ii} \in \left[\begin{array}{ccc|c} \mathcal{J} & \dots & \mathcal{I} & \\ \vdots & \ddots & \vdots & *_{m\times n-m} \\ \mathcal{I} & \dots & \mathcal{J} & \\ \hline \mathcal{I} & \dots & \mathcal{I} & \\ \vdots & \ddots & \vdots & *_{n-m\times n-m} \\ \mathcal{I} & \dots & \mathcal{I} & \end{array}\right], \qquad B^*_{ij} \in \left[\begin{array}{ccc|c} \mathcal{I} & \dots & \mathcal{I} & \\ \vdots & \ddots & \vdots & *_{m\times n-m} \\ \mathcal{I} & \dots & \mathcal{I} & \\ \hline \mathcal{I} & \dots & \mathcal{I} & \\ \vdots & \ddots & \vdots & *_{m\times n-m} \\ \mathcal{I} & \dots & \mathcal{I} & \end{array}\right].$$

These are equivalent to the existence of $\boldsymbol{B}_k$ satisfying (15) and to the fact that the corresponding entries of $B^*_{k-1}(z)$ belong to $\mathcal{I}$. $\boldsymbol{B}_k$ is obtained applying the process of reduction, a Computer Algebra algorithm, see [10]. □

Remark 3.7. For the structure of the matrix mask $\boldsymbol{B}_k$ of minimal support (in the sense of total degree) that satisfies (5) see [4, pp. 101–102]. Note that this representation of $\boldsymbol{B}_k$ is not unique. Note also that the supports of the masks $\boldsymbol{B}_k$ increase with k, if $n > 1$ and $1 \leqslant m < n$, and we have

$$\operatorname{supp}(\boldsymbol{B}_k) \subset \operatorname{supp}(\boldsymbol{A}) + [0,k]^s.$$

Now we are ready to characterize the existence of the difference schemes in terms of the structure of the polynomial eigensequences of $S_{\boldsymbol{A}}$. Denote by $\Pi_k(\mathbb{Z}^s)$ the linear span of the monomial sequences $\boldsymbol{m}_\mu = (\alpha^\mu)_{\alpha\in\mathbb{Z}^s}$, $\mu \in \mathbb{N}_0^s$, $|\mu| \leqslant k$. Denote also $\Pi_{-1}(\mathbb{Z}^s) = \{0\}$.

The result of Theorem 3.8 is similar to the one of [28, Theorem 3], where it is stated for expansive dilations and under a restrictive assumption that $\widehat{T}(\xi)$ is a similarity transformation. Our construction of $\widehat{T}(\xi)$ makes the proof of Theorem 3.8 less technical.

Theorem 3.8. *Let $\boldsymbol{B}_0 := \boldsymbol{A}$. The following statements are equivalent*:

(i) *The masks $\boldsymbol{B}_\ell \in \ell_0^{s^\ell n\times s^\ell n}(\mathbb{Z}^s)$, $k \geqslant 1$, satisfying (5) exist and, for each $\ell = 0,\dots,k-1$, the joint $2^{-\ell}$-eigenspace of the matrices*

$$B_{\ell,\varepsilon} := \sum_{\beta\in\mathbb{Z}^s} B_\ell(\varepsilon + 2\beta), \quad \varepsilon \in \{0,1\}^s,$$

has the dimension $m\cdot s^\ell$ and is spanned by the columns of

$$\{I_{s^\ell} \otimes e_j:\ j = 1,\dots,m\}. \tag{19}$$

(ii) *For $j = 1,\dots,m$ and $|\mu| \leqslant k-1$, the $2^{-|\mu|}$-eigenspaces of $S_{\boldsymbol{A}}$ satisfy*

$$S_{\boldsymbol{A}}(\boldsymbol{m}_\mu \cdot e_j) - 2^{-|\mu|}(\boldsymbol{m}_\mu \cdot e_j) \in \operatorname{span}\left\{e_i \cdot \Pi_{|\mu|-1}\left(\mathbb{Z}^s\right):\ i = 1,\dots,m\right\}. \tag{20}$$

Proof. $\Leftarrow$: The proof is by induction on k. Let $k=1$. Then by (20) we have $S_{\boldsymbol{A}}(\boldsymbol{m}_0 \cdot e_j) = \boldsymbol{m}_0 \cdot e_j$ for constant sequences $\boldsymbol{m}_0 \cdot e_j$, $j=1,\dots,m$. This implies, by definition of the subdivision operator $S_{\boldsymbol{A}}$ with $\boldsymbol{B}_0 = \boldsymbol{A}$,

$$e_j = (\boldsymbol{m}_0 \cdot e_j)(\varepsilon) = \sum_{\beta\in\mathbb{Z}^s} A(\varepsilon + 2\beta)(\boldsymbol{m}_0 \cdot e_j)(\beta) = B_{0,\varepsilon} e_j, \quad \varepsilon \in \{0,1\}^s,$$

i.e. it implies the property (19) with $k=1$. Therefore, by Theorem 3.6 there exists a difference mask $\boldsymbol{B}_1 \in \ell_0^{sn\times sn}(\mathbb{Z}^s)$ with

$$\nabla S_{\boldsymbol{A}} = S_{\boldsymbol{B}_1}\nabla.$$

By induction hypothesis with $|\mu| \leqslant k-2$ there exist $\boldsymbol{B}_{k-1} \in \ell_0^{s^{k-1}n\times s^{k-1}n}(\mathbb{Z}^s)$ satisfying (5). By (20) we get

$$\nabla^{k-1}\big(S_{\boldsymbol{A}}(\boldsymbol{m}_\mu \cdot e_j) - 2^{-k+1}\boldsymbol{m}_\mu \cdot e_j\big) = 0, \quad |\mu| = k-1,\ \mu_\ell = k-1.$$

The structure of $\boldsymbol{m}_\mu \cdot e_j$ yields that $2^{-k+1}\nabla^{k-1}(\boldsymbol{m}_\mu \cdot e_j)(\alpha)$, $\alpha \in \mathbb{Z}^s$, is one of the columns of $I_{s^{k-1}} \otimes e_j$. Thus, by definition of the operator $S_{\boldsymbol{B}_{k-1}}$, (5) and the linearity of ∇, we get that the vector $\boldsymbol{v} = \nabla^{k-1}(\boldsymbol{m}_\mu \cdot e_j)$ satisfies

$$S_{\boldsymbol{B}_{k-1}}\boldsymbol{v}(\alpha) = \nabla^{k-1}S_{\boldsymbol{A}}(\boldsymbol{m}_\mu \cdot e_j)(\alpha) = 2^{-k+1}\,\boldsymbol{v}(\alpha), \quad \alpha \in \mathbb{Z}^s.$$

Varying μ and j appropriately we get that the property (19) is satisfied for $\ell = 0,\dots,k-1$, and, thus, by Theorem 3.6 we get that there exists a difference mask $\boldsymbol{B}_k \in \ell_0^{s^k n\times s^k n}(\mathbb{Z}^s)$ satisfying (5).

$\Rightarrow$: Note that

$$S_{\boldsymbol{B}_\ell}\boldsymbol{v} = 2^{-\ell}\boldsymbol{v}, \quad \ell = 0,\dots,k-1,\ k \geqslant 1, \tag{21}$$

for any constant sequence $\boldsymbol{v} \in \ell^{ns^\ell}(\mathbb{Z}^s)$ such that all $v(\alpha)$, $\alpha \in \mathbb{Z}^s$, are equal to the same column of $I_{s^\ell} \otimes [e_1\ \dots\ e_m]$. Let $\ell = 0,\dots,k-1$. By definition of the operator ∇^ℓ, there exists $\mu \in \mathbb{N}_0^s$, $|\mu| = \ell$, and $j = 1,\dots,m$, such that $\boldsymbol{v} = \nabla^\ell(\boldsymbol{m}_\mu \cdot e_j)$. Due to identity (5) and (21), we get

$$S_{\boldsymbol{B}_\ell}\boldsymbol{v} = 2^{-\ell}\boldsymbol{v} = \nabla^\ell S_{\boldsymbol{A}}(\boldsymbol{m}_\mu \cdot e_j).$$

The linearity and definition of ∇^ℓ imply that

$$S_{\boldsymbol{A}}(\boldsymbol{m}_\mu \cdot e_j) - 2^{-\ell}\boldsymbol{m}_\mu \cdot e_j = \begin{bmatrix} \boldsymbol{p}_1 & \dots & \boldsymbol{p}_m & 0 & \dots & 0 \end{bmatrix}^T$$

for some scalar polynomial sequence $\boldsymbol{p}_j \in \Pi_{\ell-1}(\mathbb{Z}^s)$. Note that the structure of the sequence on the right-hand side is due to the definition of the operator ∇ that applied to a vector sequence in $\ell^n(\mathbb{Z}^s)$ leaves its entries $m+1,\dots,n$ unchanged. $\square$

Corollary 3.9, stated under assumptions (8)–(9), provides us with a practical tool for checking the existence of the difference schemes and follows from Theorem 3.8 and Proposition 3.1.

Corollary 3.9. *If $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ satisfies sum rules of order $k+1$ and multiplicity m, then the masks $\boldsymbol{B}_\ell \in \ell_0^{s^\ell n\times s^\ell n}(\mathbb{Z}^s)$, $1 \leqslant \ell \leqslant k+1$, satisfying* (5) *exist.*

4. Joint spectral radius versus restricted spectral radius

In this section we address the main topic of our presentation: the comparison of the JSR and the RSR. Throughout this section we assume that the eigensequences of the subdivision operator $S_{\boldsymbol{A}}$ are as in (8)–(9), implying that the joint 1-eigenspace of the submasks A_ε satisfies

$$\mathcal{E}_{\boldsymbol{A}} = \operatorname{span}\{e_1,\dots,e_m\}, \quad 1 \leqslant m \leqslant n.$$

Otherwise, we transform $\boldsymbol{A}$ using the result of Proposition 3.3.

4.1. Restricted spectral radius

For a difference scheme $S_{\boldsymbol{B}_k}$, $k \geqslant 1$, define the *restricted* (k,p)*-norm*

$$\|S_{\boldsymbol{B}_k}|_{\nabla^k}\|_p := \sup\left\{ \frac{\|S_{\boldsymbol{B}_k}\nabla^k \boldsymbol{c}\|_p}{\|\nabla^k \boldsymbol{c}\|_p} \colon \boldsymbol{c} \in \ell_p^n\big(\mathbb{Z}^s\big),\ \nabla^k \boldsymbol{c} \neq 0 \right\} \tag{22}$$

and the *restricted* (k,p)*-spectral radius* ((k,p)-RSR)

$$\rho_p(S_{\boldsymbol{B}_k}|_{\nabla^k}) := \lim_{r\to\infty} \big\|S^r_{\boldsymbol{B}_k}|_{\nabla^k}\big\|_p^{1/r}, \quad 1 \leqslant p \leqslant \infty. \tag{23}$$

The motivation for introducing the notion of the restricted spectral radius is the fact that the difference masks $\boldsymbol{B}_k$ satisfying (5) are not unique, see Remark 3.7. Thus, only the restricted spectral properties of a difference operator $S_{\boldsymbol{B}_k}$ derived from a given subdivision scheme $S_{\boldsymbol{A}}$ associated with the mask $\boldsymbol{A}$ fully characterize the convergence of $S_{\boldsymbol{A}}$, see also [5, p. 107].

Propositions 4.1 and 4.2 are crucial for the comparison of the JSR and the RSR. These results show that the RSR yields an alternative characterization of the convergence and regularity of subdivision schemes, see also Section 4.5. Furthermore, these results allow us to show in Section 4.6 that the quantity in (22) is computable, when $p=\infty$, and can be estimated in the finite number of steps, when $1 \leqslant p < \infty$.

Proposition 4.1. *Let $r \in \mathbb{N}$, $1 \leqslant k < N$ and the set $K = [-2, N]^s$. The restricted (k,∞)-norm satisfies*

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k} \big\|_\infty = \max\Big\{ \max_{\alpha \in 2^{r+1}K} \big| S^r_{\boldsymbol{B}_k} \nabla^k \boldsymbol{c}(\alpha)\big|_\infty \colon \boldsymbol{c} \in \ell^n_\infty(-2K),\ \big\|\nabla^k \boldsymbol{c}|_{-2K}\big\|_\infty = 1\Big\}.$$

Proof. The definition of the restricted (k,∞)-norm is equivalent to

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k} \big\|_\infty = \max\Big\{ \max_{\alpha \in \mathbb{Z}^s} \big| S^r_{\boldsymbol{B}_k} \nabla^k \boldsymbol{c}(\alpha)\big|_\infty \colon \boldsymbol{c} \in \ell^n_\infty\big(\mathbb{Z}^s\big),\ \big\|\nabla^k \boldsymbol{c}\big\|_\infty = 1\Big\}.$$

Using the definition and periodicity of $S_{\boldsymbol{B}_k}$, we get

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k} \big\|_\infty = \max\left\{ \max_{\alpha \in [0,2^r-1]^s \cap \mathbb{Z}^s} \left| \sum_{\beta \in \mathbb{Z}^s} B_k^{(r)}\big(\alpha - 2^r\beta\big) \nabla^k \boldsymbol{c}(\beta)\right|_\infty \colon \boldsymbol{c} \in \ell^n_\infty\big(\mathbb{Z}^s\big),\ \big\|\nabla^k \boldsymbol{c}\big\|_\infty = 1\right\}$$

with the iterated masks $\boldsymbol{B}_k^{(r)}$ are defined as in (3). Due to the compact support of $\boldsymbol{B}_k$ we have

$$\alpha - 2^r\beta \in 2^r[0, N+k]^s \quad \Rightarrow \quad \beta \in [-N-k, 1]^s.$$

This, for fixed $\alpha \in [0, 2^r-1]^s \cap \mathbb{Z}^s$, leads to

$$\left| \sum_{\beta \in \mathbb{Z}^s} B_k^{(r)}\big(\alpha - 2^r\beta\big) \nabla^k \boldsymbol{c}(\beta)\right|_\infty = \left| \sum_{\beta \in [-N-k,1]^s} B_k^{(r)}\big(\alpha - 2^r\beta\big) \nabla^k \boldsymbol{c}(\beta)\right|_\infty.$$

This also implies that, to compute the (k,∞)-restricted norm, it is enough to take the maximum over the sequences $\boldsymbol{c} \in \ell^n_\infty(-2K)$ with $\|\nabla^k \boldsymbol{c}|_{-2K}\|_\infty = 1$. Computing the maximum over $\alpha \in 2^{r+1}K$ does not change the value of the (k,∞)-restricted norm, due to the periodicity of the operator $S^r_{\boldsymbol{B}_k}$: if, due to the restriction $\beta \in [-N-k, 1]^s$, the sum

$$\sum_{\beta \in [-N-k,1]^s} B_k^{(r)}\big(\alpha - 2^r\beta\big) \nabla^k \boldsymbol{c}(\beta) \tag{24}$$

does not include all non-zero matrices $B_k^{(r)}(\alpha - 2^r\beta)$ from a particular coset determined by $\alpha \in 2^{r+1}K$, then we just interpret the value we get in (24) as the one obtained for some $\alpha \in [0, 2^r-1]^s$ and the particular choice of $\boldsymbol{c} \in \ell^n_\infty(-2K)$ with zeros at the positions corresponding to the missing matrices $B_k^{(r)}(\alpha - 2^r\beta)$. $\square$

Proposition 4.2. *Let $r \in \mathbb{N}$, $1 \leqslant k < N$, $1 \leqslant p < \infty$, $K = [-2, N]^s$, $\widetilde{K} = [-2N-4, 2N+4]^s$ and*

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k,1} \big\|_p^p := \max_{\substack{\boldsymbol{c} \in \ell^n_p(\widetilde{K}) \\ \|\nabla^k \boldsymbol{c}|_{\widetilde{K}}\|_p = 1}} \left\{ \sum_{\alpha \in 2^{r+1}K} \big| S^r_{\boldsymbol{B}_k} \nabla^k \boldsymbol{c}(\alpha)\big|_p^p \right\},$$

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k,2} \big\|_p^p := \max_{\substack{\boldsymbol{c} \in \ell^n_p(\widetilde{K}) \\ \|\nabla^k \boldsymbol{c}\|_p = 1}} \left\{ \sum_{\alpha \in 2^{r+1}K} \big| S^r_{\boldsymbol{B}_k} \nabla^k \boldsymbol{c}(\alpha)\big|_p^p \right\}.$$

The restricted (k,p)-norm satisfies

$$\big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k,2} \big\|_p \leqslant \big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k} \big\|_p \leqslant 3 \sum_{\ell=0}^{s-1} 2^{s-\ell} \binom{s}{\ell} \big\| S^r_{\boldsymbol{B}_k}|_{\nabla^k,1} \big\|_p.$$

Proof. Note that by the definition of K we have

$$\bigcup_{\gamma \in \mathbb{Z}^s} 2^{r+1}K + 2^{r+1}(N+2)\gamma = \mathbb{Z}^s.$$

Then, for any $\alpha \in \mathbb{Z}^s$, there exist $\alpha' \in 2^{r+1}K$ and $\gamma \in \mathbb{Z}^s$ such that $\alpha = \alpha' + 2^{r+1}(N+2)\gamma$. Let $\boldsymbol{c} \in \ell_p^n(\mathbb{Z}^s)$. By (1) and the definition of the operator $S_{\boldsymbol{B}_k}$ we get

$$\begin{aligned}\left\| S_{\boldsymbol{B}_k}^r \nabla^k \boldsymbol{c} \right\|_p^p &= \sum_{\alpha \in \mathbb{Z}^s} \left| \sum_{\beta \in \mathbb{Z}^s} B_k^{(r)}\big(\alpha - 2^r\beta\big)\nabla^k c(\beta)\right|_p^p \\ &\leqslant \sum_{\gamma \in \mathbb{Z}^s} \sum_{\alpha' \in 2^{r+1}K} \left| \sum_{\beta \in \mathbb{Z}^s} B_k^{(r)}\big(\alpha' + 2^{r+1}(N+2)\gamma - 2^r\beta\big)\nabla^k c(\beta)\right|_p^p.\end{aligned}$$

By the compact support of $\boldsymbol{B}_k$, we get

$$\alpha' + 2^{r+1}(N+2)\gamma - 2^r\beta \in 2^{r+1}K \quad \Leftrightarrow \quad \beta \in 2^{-r}\alpha' + (2N+4)\gamma - 2K.$$

The fact that $\alpha' \in 2^{r+1}K$ implies that for fixed γ it suffices to consider $\beta \in \widetilde{K} + (2N+4)\gamma$. Let $\beta' = \beta - (2N+4)\gamma$. We get

$$\left\| S_{\boldsymbol{B}_k}^r \nabla^k \boldsymbol{c} \right\|_p^p \leqslant \sum_{\gamma \in \mathbb{Z}^s} \sum_{\alpha' \in 2^{r+1}K} \left| \sum_{\beta' \in \widetilde{K}} B_k^{(r)}\big(\alpha' - 2^r\beta'\big)\nabla^k c\big(\beta' + (2N+4)\gamma\big)\right|_p^p.$$

The definition of $\|S_{\boldsymbol{B}_k}^r|_{\nabla^k,1}\|_p$ yields

$$\left\| S_{\boldsymbol{B}_k}^r \nabla^k \boldsymbol{c} \right\|_p^p \leqslant \sum_{\gamma \in \mathbb{Z}^s} \left\| S_{\boldsymbol{B}_k}^r|_{\nabla^k,1} \right\|_p^p \left\| \nabla^k \boldsymbol{c}|_{\widetilde{K}+(2N+4)\gamma} \right\|_p^p.$$

Note that the sets $\widetilde{K} + (2N+4)\gamma$, $\gamma \in \mathbb{Z}^s$, intersect. This yields

$$\sum_{\gamma \in \mathbb{Z}^s} \left\| \nabla^k \boldsymbol{c}|_{\widetilde{K}+(2N+4)\gamma} \right\|_p^p \leqslant 3 \sum_{\ell=0}^{s-1} 2^{s-\ell} \binom{s}{\ell} \left\| \nabla^k \boldsymbol{c} \right\|_p^p.$$

The reverse inequality

$$\left\| S_{\boldsymbol{B}_k}^r|_{\nabla^k} \right\|_p \geqslant \left\| S_{\boldsymbol{B}_k}^r|_{\nabla^k,2} \right\|_p$$

follows easily as the supremum in (22) is taken over a larger space. □

Remark 4.3.

(1) The choice of $\alpha \in 2^{r+1}K$ in Propositions 4.1 and 4.2 is important for the proof of Propositions 4.6 and 4.10. Otherwise, it suffices to consider $\alpha \in [0, 2^r - 1]^s$ when computing the (k, ∞)-restricted norm.
(2) The compact support of the masks $\boldsymbol{B}_k$, $k = 1, \ldots, N-1$, allows for other equivalent ways of writing the (k, ∞)-restricted norm in (22), see [4] and [5]. In the case $1 \leqslant p < \infty$, the result of Proposition 4.2 improve the corresponding result in [5] for $k = 0$.

4.2. Joint spectral radius

Let us first recall the definition of the p-norm *joint spectral radius* of a finite collection of matrices, see for example [21]. Denote by $\mathcal{A}_\varepsilon$ the linear operators on $\ell_0^{1\times n}(\mathbb{Z}^s)$ given by

$$\mathcal{A}_\varepsilon \boldsymbol{v}(\alpha) := \sum_{\beta \in \mathbb{Z}^s} v(\beta)A(\varepsilon + 2\alpha - \beta), \quad \varepsilon \in \{0,1\}^s,\ \alpha \in \mathbb{Z}^s. \tag{25}$$

$\mathcal{A}_{(0,\ldots,0)}$ is sometimes called the transition operator associated with $\boldsymbol{A}$ and is the algebraic adjoint of $S_{\boldsymbol{A}}$ with respect to the bilinear form

$$\langle \boldsymbol{v}, \boldsymbol{u} \rangle := \sum_{\alpha \in \mathbb{Z}^s} v(\alpha)u(-\alpha), \quad \boldsymbol{u} \in \ell^n\big(\mathbb{Z}^s\big),\ \boldsymbol{v} \in \ell_0^{1\times n}\big(\mathbb{Z}^s\big).$$

Let $\mathcal{A}$ be a finite collection of the form

$$\mathcal{A} := \big\{\mathcal{A}_\varepsilon \colon \varepsilon \in \{0,1\}^s\big\}.$$

To reduce $\mathcal{A}$ to a finite collection of matrices we proceed as follows. The definition in (25) implies that $\ell^{1\times n}(K)$ is invariant under all $\mathcal{A}_\varepsilon$, $\varepsilon \in \{0,1\}^s$, i.e. $\mathcal{A}_\varepsilon \boldsymbol{v} \in \ell^{1\times n}(K)$ for any $\varepsilon \in \{0,1\}^s$ and $\boldsymbol{v} \in \ell^{1\times n}(K)$. The latter follows by the fact that there is a non-zero entry in (25) if and only if $\varepsilon + 2\alpha - \beta \in K$ and $\beta \in K$. And, hence,

$$\alpha \in \frac{1}{2}(K - \varepsilon + K) = \left\{K - \frac{\varepsilon}{2}\right\} \cap \mathbb{Z}^s = K \cap \mathbb{Z}^s.$$

A block matrix representation of $\mathcal{A}_\varepsilon : \ell^{1\times n}(K) \to \ell^{1\times n}(K)$ is of the form

$$\left[A^T(\varepsilon + 2\alpha - \beta)\right]_{\alpha,\beta\in K}, \quad \varepsilon \in \{0,1\}^s. \tag{26}$$

Let V be a finite-dimensional subspace of $\ell^{1\times n}(K)$, which is invariant under all $\mathcal{A}_\varepsilon$. The invariance of $\ell^{1\times n}(K)$ under $\mathcal{A}_\varepsilon$ implies that the restrictions of $\mathcal{A}_\varepsilon$ to V are well defined. Denote by

$$\mathcal{A}|_V := \left\{\mathcal{A}_\varepsilon|_V \colon \varepsilon \in \{0,1\}^s\right\}$$

the finite collection of matrix representations $\mathcal{A}_\varepsilon|_V$ of $\mathcal{A}_\varepsilon$ with respect to a basis of V. For examples of such matrix representations see [17, Example 3.5]. For each $1 \leqslant p \leqslant \infty$, the *joint spectral radius* (p-JSR) of $\mathcal{A}|_V$ is defined by

$$\rho_p(\mathcal{A}|_V) := \begin{cases} \lim_{r\to\infty}\left(\sum_{\varepsilon_1,\dots,\varepsilon_r\in\{0,1\}^s} |\mathcal{A}_{\varepsilon_1}|_V \cdot \dots \cdot \mathcal{A}_{\varepsilon_r}|_V|^p\right)^{1/rp}, & p < \infty, \\ \lim_{r\to\infty}\max_{\varepsilon_1,\dots,\varepsilon_r\in\{0,1\}^s} |\mathcal{A}_{\varepsilon_1}|_V \cdot \dots \cdot \mathcal{A}_{\varepsilon_r}|_V|^{1/r}, & p = \infty, \end{cases}$$

the limit above exists and is independent of the choice of the matrix norm $|\cdot|$. For computational aspects of the joint spectral radius see [27] and references therein. The choice of V, as for example in (27) and (36), is of great importance. Firstly, its dimension determines the size of the matrices in (26) and should be preferably small. Secondly, V should be such as to allow us to connect the regularity of a subdivision scheme with the estimates for (k,p)-JSR. For a better insight into such a connection see the survey [1].

4.3. Scalar case

In the scalar case we have $\boldsymbol{A} \in \ell_0(\mathbb{Z}^s)$. The first and crucial step for comparing p-JSR and (k,p)-RSR is to study the structure of the following subspaces of $\ell(\mathbb{Z}^s)$: For $K = [-2,N]^s$ and $k \in \mathbb{N}_0$, $0 \leqslant k < N$, we define the shift-invariant sequence spaces

$$\begin{aligned} U_k &:= \operatorname{span}\left\{\boldsymbol{x}_\mu(\cdot - \beta)\colon \beta \in \mathbb{Z}^s,\ |\mu| \leqslant k\right\} = \Pi_k\left(\mathbb{Z}^s\right) \quad \text{and} \\ V_k &:= \left\{\boldsymbol{v} \in \ell(K)\colon \sum_{\beta\in\mathbb{Z}^s} v(\beta)u(-\beta) = 0 \text{ for all } \boldsymbol{u} \in U_k\right\} \end{aligned} \tag{27}$$

with the sequences $\boldsymbol{x}_\mu \in \ell(\mathbb{Z}^s)$ being the polynomial eigensequences of $S_{\boldsymbol{A}}$ in (8)–(9). The spaces U_k and V_k appear also in [6, Section 3]. Part (iii) of Lemma 4.4 is the key to understanding the relation between the joint and the restricted spectral radii in the scalar case.

Lemma 4.4. *Let* $0 \leqslant k < N$.

(i) *U_k is invariant under the subdivision operator $S_{\boldsymbol{A}}$.*
(ii) *V_k is invariant under $\mathcal{A}_\varepsilon$, $\varepsilon \in \{0,1\}^s$.*
(iii)

$$V_k = \operatorname{span}\left\{\left(\widetilde{\nabla}^\mu \boldsymbol{\delta}\right)(\cdot - \beta)\colon \beta \in K,\ \mu \in \mathbb{N}_0^s,\ |\mu| = k+1\right\} \cap \ell(K). \tag{28}$$

Proof. (i): The claim follows by the properties of $S_{\boldsymbol{A}}$ and the structure of its polynomial $2^{-|\mu|}$-eigenspaces, $|\mu| \leqslant k$.

(ii): Let $\boldsymbol{v} \in V_k$ and $\varepsilon \in \{0,1\}^s$. Then, for any $\boldsymbol{u} \in U_k$, by (25), we get

$$\sum_{\alpha\in\mathbb{Z}^s} \mathcal{A}_\varepsilon \boldsymbol{v}(\alpha)u(-\alpha) = \sum_{\alpha,\beta\in\mathbb{Z}^s} v(\beta)A(\varepsilon + 2\alpha - \beta)u(-\alpha).$$

By (i), $(S_{\boldsymbol{A}}\boldsymbol{u})(\cdot + \varepsilon) \in U_k$. And, thus, by definition of $S_{\boldsymbol{A}}$ and V_k, we obtain

$$\sum_{\beta\in\mathbb{Z}^s} v(\beta) \sum_{\alpha\in\mathbb{Z}^s} A(\varepsilon - \beta + 2\alpha)u(-\alpha) = \sum_{\beta\in\mathbb{Z}^s} v(\beta)\left((S_{\boldsymbol{A}}\boldsymbol{u})(\cdot + \varepsilon)\right)(-\beta) = 0.$$

(iii): The proof is by induction on k. Case $k = 0$ is discussed in [5, Lemma 2]. Assume that (28) holds for some $k \in \mathbb{N}$. We show, next that (28) holds for $k+1$. Take any $\boldsymbol{v} \in V_{k+1}$. By (27) we have

$$\sum_{\beta\in\mathbb{Z}^s} v(\beta)u(-\beta) = 0 \quad \text{for all } \boldsymbol{u} \in U_{k+1},$$

which implies, due to $U_k \subset U_{k+1}$, that $\sum_{\beta\in\mathbb{Z}^s} v(\beta)u(-\beta) = 0$ for all $\boldsymbol{u} \in U_k$. Thus, by the induction hypothesis the sequence $\boldsymbol{v}$ is of the form

$$\boldsymbol{v} = \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\widetilde{\nabla}^\mu \boldsymbol{\delta}(\cdot - \beta) \tag{29}$$

with $\boldsymbol{v}_\mu$ being some sequences supported on K. Then, by definition (27) of V_{k+1} and (29), we get

$$\begin{aligned}
0 = \sum_{\alpha\in\mathbb{Z}^s} v(\alpha)u(-\alpha) &= \sum_{\alpha\in\mathbb{Z}^s}\Biggl(\sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha - \beta)\Biggr)u(-\alpha) \\
&= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\Biggl(\sum_{\alpha\in\mathbb{Z}^s}\widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha - \beta)u(-\alpha)\Biggr)
\end{aligned}$$

for any $\boldsymbol{u} \in U_{k+1}$. Due to the following property of the difference operator:

$$\widetilde{\nabla}^\mu \boldsymbol{c} = \widetilde{\nabla}^\mu\Biggl(\sum_{\alpha\in\mathbb{Z}^s} \boldsymbol{\delta}(\cdot - \alpha)c(\alpha)\Biggr) = \sum_{\alpha\in\mathbb{Z}^s} \widetilde{\nabla}^\mu \boldsymbol{\delta}(\cdot - \alpha)c(\alpha), \quad \boldsymbol{c} \in \ell\bigl(\mathbb{Z}^s\bigr), \tag{30}$$

we have, for any $\boldsymbol{u} \in U_{k+1}$,

$$\sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\Biggl(\sum_{\alpha\in\mathbb{Z}^s}\widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha - \beta)u(-\alpha)\Biggr) = \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\bigl(\widetilde{\nabla}^\mu \boldsymbol{u}\bigr)(-\beta).$$

Choose $\boldsymbol{u} = \boldsymbol{x}_{\tilde\mu} \in U_{k+1}$, $\tilde\mu \in \mathbb{N}_0^s$, $|\tilde\mu| = k + 1$. Then

$$\sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s} v_\mu(\beta)\bigl(\widetilde{\nabla}^\mu \boldsymbol{x}_{\tilde\mu}\bigr)(-\beta) = \sum_{\alpha\in\mathbb{Z}^s} v_{\tilde\mu}(\alpha) = 0.$$

Thus, because $\tilde\mu$ is arbitrary, all of the Laurent polynomials

$$v_\mu^*(z) = 2^{-s}\sum_{\alpha\in K} v_\mu(\alpha)z^\alpha, \quad \mu \in \mathbb{N}_0^s,\ |\mu| = k + 1,$$

are in the ideal $\langle 1 - z_\ell\colon\ \ell = 1, \ldots, s\rangle$ of polynomials that vanish at 1 due to $v_\mu^*(1) = 0$. Therefore,

$$v_\mu^*(z) = \sum_{\ell=1}^{s}(1 - z_\ell)v_{\mu,\ell}^*(z) = \sum_{\ell=1}^{s}(\nabla_\ell \boldsymbol{v}_{\mu,\ell})^*(z)$$

with the sequences $\boldsymbol{v}_{\mu,\ell}$ supported on K. Thus, by (30),

$$\boldsymbol{v}_\mu = \sum_{\ell=1}^{s}\sum_{\beta\in\mathbb{Z}^s} v_{\mu,\ell}(\beta)\nabla_\ell \boldsymbol{\delta}(\cdot - \beta). \tag{31}$$

By induction assumption (29), (30) and (31) we get

$$\begin{aligned}
\boldsymbol{v} &= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\alpha\in\mathbb{Z}^s}\Biggl(\sum_{\ell=1}^{s}\sum_{\beta\in\mathbb{Z}^s} v_{\mu,\ell}(\beta)\nabla_\ell \boldsymbol{\delta}(\alpha - \beta)\Biggr)\bigl(\widetilde{\nabla}^\mu \boldsymbol{\delta}\bigr)(\cdot - \alpha) \\
&= \sum_{\substack{\mu\in\mathbb{N}_0^s \\ |\mu|=k+1}} \sum_{\beta\in\mathbb{Z}^s}\sum_{\ell=1}^{s} v_{\mu,\ell}(\beta)\widetilde{\nabla}^{\mu+\ell} \boldsymbol{\delta}(\cdot - \beta). \qquad \square
\end{aligned}$$

Note that part (iii) of Lemma 4.4 states that V_k is finite-dimensional and leads to the following equivalent formulation of the joint spectral radius.

Lemma 4.5. *Let $0 \leqslant k < N$. Assume that $\boldsymbol{A} \in \ell_0(\mathbb{Z}^s)$ with $\operatorname{supp}(\boldsymbol{A}) \subset [0,N]^s$ satisfies sum rules of order $k+1$ and multiplicity 1. Then, for V_k in (27), we have*

$$\rho_p(\mathcal{A}|_{V_k}) = \lim_{r\to\infty} \max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \big\|\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}\big\|_p^{1/r}, \quad 1 \leqslant p \leqslant \infty.$$

Proof. Let $r \in \mathbb{N}_0$. By part (iii) of Lemma 4.4, there exists a positive constant $C_1 > 0$ independent of r and such that

$$\big\|\mathcal{A}^r|_{V_k}\big\|_p \leqslant C_1 \cdot \max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \big\|\mathcal{A}^r\widetilde{\nabla}^\mu \boldsymbol{\delta}\big\|_p.$$

By [6, p. 144] and the definition of $\mathcal{A}$,

$$\big\|\mathcal{A}^r\widetilde{\nabla}^\mu \boldsymbol{\delta}\big\|_p = \bigg\|\sum_{\alpha\in\mathbb{Z}^s} \widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha) A^{(r)}(\cdot-\alpha)\bigg\|_p, \quad \mu\in\mathbb{N}_0^s,\ |\mu|=k+1.$$

Thus, (30), the fact that $\boldsymbol{A}^{(r)} = S_{\boldsymbol{A}}^r \boldsymbol{\delta}$ and the definition of $\boldsymbol{\delta}$ yield

$$\bigg\|\sum_{\alpha\in\mathbb{Z}^s} \widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha) A^{(r)}(\cdot-\alpha)\bigg\|_p = \bigg\|\sum_{\alpha\in\mathbb{Z}^s} \delta(\alpha)\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}(\cdot-\alpha)\bigg\|_p = \big\|\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}\big\|_p.$$

Therefore, we obtain

$$\rho_p(\mathcal{A}|_{V_k}) \leqslant \lim_{r\to\infty} \max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \big\|\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}\big\|_p^{1/r}.$$

To show the reverse inequality, observe first that by definition of $\|\mathcal{A}^r\boldsymbol{v}\|_p$ and $\mathcal{A}^r|_{V_k}$ we get

$$\big\|\mathcal{A}^r|_{V_k}\big\|_p \geqslant \max\big\{\big\|\mathcal{A}^r\boldsymbol{v}\big\|_p\colon\ \boldsymbol{v}\in V_k,\ \|\boldsymbol{v}\|_p = 1\big\}.$$

Due to the compact support of $\boldsymbol{\delta}$ we have

$$1 \leqslant \big\|\widetilde{\nabla}^\mu \boldsymbol{\delta}\big\|_p =: C_2 < \infty.$$

Note also that $\widetilde{\nabla}^\mu \boldsymbol{\delta} \in V_k$. Thus, by definition of $\|\mathcal{A}^r|_{V_k}\|_p$, we get

$$\big\|\mathcal{A}^r|_{V_k}\big\|_p \geqslant C_2^{-1}\big\|\mathcal{A}^r\widetilde{\nabla}^\mu \boldsymbol{\delta}\big\|_p, \quad \mu\in\mathbb{N}_0^s,\ |\mu|=k+1.$$

The same argument as above, implies

$$\rho_p(\mathcal{A}|_{V_k}) \geqslant \lim_{r\to\infty} \max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \big\|\widetilde{\nabla}^\mu S_{\boldsymbol{A}} \boldsymbol{\delta}\big\|_p^{1/r}. \qquad \square$$

Now we are ready for the main result of this subsection. Its proof for the case $k=0$ is given in [5, Proposition 1].

Proposition 4.6. *Let $1 \leqslant p \leqslant \infty$ and $0 \leqslant k < N$. Assume that $\boldsymbol{A} \in \ell_0(\mathbb{Z}^s)$ with $\operatorname{supp}(\boldsymbol{A}) \subset [0,N]^s$ satisfies the sum rules of order $k+1$ and multiplicity 1, V_k as in (27) and $\boldsymbol{B}_{k+1} \in \ell_0^{s^{k+1}\times s^{k+1}}(\mathbb{Z}^s)$ satisfies (5). Then*

$$\rho_p(S_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}) = \rho_p(\mathcal{A}|_{V_k}).$$

Proof. Let $r \in \mathbb{N}$. By Lemma 4.5 and (23), it suffices to show that the quantities $\|S_{\boldsymbol{B}_{k+1}}^r|_{\nabla^{k+1}}\|_p$ and $\max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \|\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}\|_p$ are equivalent. We start with the case $p=\infty$ and show first that

$$\big\|S_{\boldsymbol{B}_{k+1}}^r|_{\nabla^{k+1}}\big\|_\infty \geqslant \binom{k+1}{\lfloor\frac{k+1}{2}\rfloor}^{-1} \max_{\substack{\mu\in\mathbb{N}_0^s\\ |\mu|=k+1}} \big\|\widetilde{\nabla}^\mu S_{\boldsymbol{A}}^r \boldsymbol{\delta}\big\|_\infty. \tag{32}$$

Let $\mu\in\mathbb{N}_0^s$, $|\mu|=k+1$. By definition of $\widetilde{\nabla}^\mu$ and $\boldsymbol{\delta}$, we have

$$1 \leqslant \big\|\widetilde{\nabla}^\mu \boldsymbol{\delta}\big\|_\infty \leqslant \binom{k+1}{\lfloor\frac{k+1}{2}\rfloor} =: C_1. \tag{33}$$

Thus, due to Proposition 4.1 and (5), we get

$$\left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_\infty \geqslant C_1^{-1} \max_{\alpha \in 2^{r+1}K} \left| \nabla^{k+1} S^r_{\boldsymbol{A}} \boldsymbol{\delta}(\alpha) \right|_\infty, \quad K = [-2, N]^s.$$

The application of ∇^{k+1} to the matrix sequence $S^r_{\boldsymbol{A}} \boldsymbol{\delta} = \boldsymbol{A}^{(r)}$ increases its support by $k+1$ in each coordinate direction and we have

$$\operatorname{supp} \nabla^{k+1} \left(S^r_{\boldsymbol{A}} \boldsymbol{\delta} \right) \subset \left(2^r - 1 \right) K + [0, k+1]^s \subset 2^{r+1} K, \quad r \in \mathbb{N}.$$

Therefore, by

$$\left\| \nabla^{k+1} \boldsymbol{c} \right\|_\infty = \left\| \begin{bmatrix} \widetilde{\nabla}^{\mathcal{M}_{k+1}(1)} \\ \vdots \\ \widetilde{\nabla}^{\mathcal{M}_{k+1}(s^k)} \end{bmatrix} \cdot \boldsymbol{c} \right\|_\infty = \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left\| \widetilde{\nabla}^\mu \boldsymbol{c} \right\|_\infty, \tag{34}$$

we obtain

$$\max_{\alpha \in 2^{r+1}K} \left| \nabla^{k+1} S^r_{\boldsymbol{A}} \boldsymbol{\delta}(\alpha) \right|_\infty = \left\| \nabla^{k+1} S^r_{\boldsymbol{A}} \boldsymbol{\delta} \right\|_\infty = \max_{\substack{mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} \right\|_\infty.$$

Thus (32) holds. We show next that there exists a constant $C_2 > 0$ such that

$$\left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_\infty \leqslant C_2 \cdot \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} \right\|_\infty.$$

Let $\boldsymbol{c} \in \ell_\infty(-2K)$ be a maximizing sequence such that

$$\left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_\infty = \max_{\alpha \in 2^{r+1}K} \left| S^r_{\boldsymbol{B}_{k+1}} \nabla^{k+1} \boldsymbol{c}(\alpha) \right|_\infty, \quad \left\| \nabla^{k+1} \boldsymbol{c}|_{-2K} \right\|_\infty = 1.$$

Since, the sequence $\boldsymbol{c}$ has compact support and $\|\nabla^{k+1} \boldsymbol{c}|_{-2K}\|_\infty = 1$, it follows that $\|\boldsymbol{c}\|_\infty$ is bounded, i.e., there exists a constant $C_3 > 0$ such that

$$\|\boldsymbol{c}\|_\infty = \max_{\beta \in \mathbb{Z}^s} \left| c(\beta) \right|_\infty = \max_{\beta \in -2K} \left| c(\beta) \right|_\infty \leqslant C_3. \tag{35}$$

Hence, by (30) and (35), we have

$$\begin{aligned} \left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_\infty &= \left\{ \max_{\alpha \in 2^{r+1}K} \left| \left[S^r_{\boldsymbol{B}_{k+1}} \sum_{\beta \in -2K} \nabla^{k+1} \boldsymbol{\delta}(\cdot - \beta) c(\beta) \right] (\alpha) \right|_\infty \right\} \\ &\leqslant C_3 \sum_{\beta \in -2K} \max_{\alpha \in 2^{r+1}K} \left| S^r_{\boldsymbol{B}_{k+1}} \nabla^{k+1} \boldsymbol{\delta}(\alpha - \beta) \right|_\infty. \end{aligned}$$

The sum above is finite and $\operatorname{supp}(S^r_{\boldsymbol{B}_{k+1}} \nabla^{k+1} \boldsymbol{\delta}) = \operatorname{supp}(\nabla^{k+1} S^r_{\boldsymbol{A}} \boldsymbol{\delta}) \subset 2^{r+1} K$. Thus, we get for $C := C_3 \cdot |2K|$

$$\left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_\infty \leqslant C \cdot \max_{\alpha \in 2^{r+1}K} \left| S^r_{\boldsymbol{B}_{k+1}} \nabla^{k+1} \boldsymbol{\delta}(\alpha) \right|_\infty = C \cdot \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} \right\|_\infty.$$

The proof for $1 \leqslant p < \infty$ is similar. We give only the steps that are different. Note that $\operatorname{supp}(\nabla^{k+1} \boldsymbol{\delta}) \subset [0, k+1]^s \subset K$ and, thus,

$$1 \leqslant \left\| \nabla^{k+1} \boldsymbol{\delta} \right\|_p = \left\| \nabla^{k+1} \boldsymbol{\delta}|_K \right\|_p = \left(\sum_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \sum_{\alpha \in K} \left| \widetilde{\nabla}^\mu \boldsymbol{\delta}(\alpha) \right|^p \right)^{1/p} =: \widetilde{C}_1 < \infty.$$

Then by Proposition 4.2, the fact that $\operatorname{supp}(\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta}) \subset 2^{r+1} K$ and by

$$\left\| \nabla^{k+1} \boldsymbol{d} \right\|_p \geqslant \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left(\sum_{\alpha \in \mathbb{Z}^s} \left| \widetilde{\nabla}^\mu \boldsymbol{d}(\alpha) \right|_p^p \right)^{1/p} = \max_{\substack{\mu \in \mathbb{N}_0^s \\ |\mu| = k+1}} \left\| \widetilde{\nabla}^\mu \boldsymbol{d} \right\|_p, \quad \boldsymbol{d} \in \ell_p\left(\mathbb{Z}^s\right),$$

we get

$$\begin{aligned}\left\| S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}} \right\|_p &\geqslant \frac{1}{\widetilde{C}_1} \max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}} \left(\sum_{\alpha\in 2^{r+1}K} \left|\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}}\boldsymbol{\delta}(\alpha)\right|^p \right)^{1/p}\\ &= \frac{1}{\widetilde{C}_1} \max_{\substack{\mu\in\mathbb{N}_0^s\\|\mu|=k+1}} \left\|\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}}\boldsymbol{\delta}\right\|_p.\end{aligned}$$

The reverse inequality is obtained as in the case $p=\infty$. $\square$

4.4. Vector case

Similar to the scalar case, we define U_k by

$$U_k = \operatorname{span}\left\{\boldsymbol{x}_{j,\mu}(\cdot-\beta)\colon\ j=1,\ldots,m,\ \beta\in\mathbb{Z}^s,\ \mu\in\mathbb{N}_0^s,\ |\mu|\leqslant k\right\} = \begin{pmatrix} \Pi_k(\mathbb{Z}^s)\\ \vdots\\ \Pi_k(\mathbb{Z}^s)\\ 0_{n-m\times 1}\end{pmatrix}$$

with $\boldsymbol{x}_{j,\mu}\in\ell^n(\mathbb{Z}^s)$ being the eigensequences of $S_{\boldsymbol{A}}$ satisfying (8)–(9). We also define for $K=[-2,N]^s$

$$V_k = \left\{\boldsymbol{v}\in\ell^{1\times n}(K)\colon \sum_{\beta\in\mathbb{Z}^s} v(\beta)u(-\beta)=0 \text{ for all } \boldsymbol{u}\in U_k\right\}. \tag{36}$$

As in the scalar case, the structure of V_k is of great importance. It is completely determined by the structure of the eigensequences of $S_{\boldsymbol{A}}$ as described by part (iii) of Lemma 4.7.

Lemma 4.7. *Let* $0\leqslant k<N$.

(i) *U_k is invariant under the subdivision operator $S_{\boldsymbol{A}}$.*
(ii) *V_k is invariant under $\mathcal{A}_\varepsilon$, $\varepsilon\in\{0,1\}^s$.*
(iii)

$$V_k = \operatorname{span}\left\{\left(\widetilde{\nabla}^\mu\boldsymbol{\delta}e_j\right)^T(\cdot-\beta)\colon\ \beta\in K,\ \mu\in\mathbb{N}_0^s,\ |\mu|=k+1,\ 1\leqslant j\leqslant n\right\}\cap\ell^{1\times n}(K).$$

Proof. (i) and (ii): the proof is as in the scalar case. (iii): the proof is the straightforward extension of [5, Lemma 3 Part 2)]. $\square$

One of the main differences between the scalar and vector cases is that the result of Lemma 4.5 cannot be reproduced in the vector case. Using the same technique as in Lemma 4.5 we obtain the following result.

Lemma 4.8. *Let $1\leqslant p\leqslant\infty$ and $0\leqslant k<N$. Assume that $\boldsymbol{A}\in\ell_0^{n\times n}(\mathbb{Z}^s)$ with $\operatorname{supp}(\boldsymbol{A})\subset[0,N]^s$ satisfies the sum rules of order $k+1$ and multiplicity m, $1\leqslant m\leqslant n$. Then, for V_k in* (36)*, we have*

$$\rho_p(\mathcal{A}|_{V_k}) = \lim_{r\to\infty} \max_{\substack{\mu\in\mathbb{N}_0^s,\ |\mu|=k+1\\ 1\leqslant j\leqslant n}} \left\|e_j^T\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}}\boldsymbol{\delta}I_n\right\|_p^{1/r}.$$

Proof. Let $r\in\mathbb{N}$ and $\varepsilon_1,\ldots,\varepsilon_r\in\{0,1\}^s$. By part (iii) of Lemma 4.7 and due to $\|\boldsymbol{v}\|_p=1$, we have for some constant $C_1>0$ independent of r

$$\left\|\mathcal{A}^r\boldsymbol{v}\right\|_p := \|\mathcal{A}_{\varepsilon_1}\ldots\mathcal{A}_{\varepsilon_r}\boldsymbol{v}\|_p \leqslant C_1\cdot \max_{\substack{\mu\in\mathbb{N}_0^s,\ |\mu|=k+1\\ 1\leqslant j\leqslant n}} \left\|\mathcal{A}^r\left(\widetilde{\nabla}^\mu\boldsymbol{\delta}e_j\right)^T\right\|_p, \quad \boldsymbol{v}\in V_k.$$

By [6, (4.1)] and definition of $\mathcal{A}$,

$$\left\|\mathcal{A}^r\left(\widetilde{\nabla}^\mu\boldsymbol{\delta}e_j\right)^T\right\|_p = \left\|\sum_{\alpha\in\mathbb{Z}^s}\left(\widetilde{\nabla}^\mu\boldsymbol{\delta}e_j\right)^T(\alpha)A^{(r)}(\cdot-\alpha)\right\|_p.$$

Thus, (30), the fact that $\boldsymbol{A}^{(r)} = S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n$ and the definition of $\boldsymbol{\delta}$ yield

$$\left\| \sum_{\alpha \in \mathbb{Z}^s} \left(\widetilde{\nabla}^\mu \boldsymbol{\delta} e_j\right)^T(\alpha) A^{(r)}(\cdot - \alpha) \right\|_p = \left\| \sum_{\alpha \in \mathbb{Z}^s} \left(\boldsymbol{\delta} e_j^T\right)(\alpha) \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n(\cdot - \alpha) \right\|_p$$
$$= \left\| e_j^T \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n \right\|_p.$$

Therefore, we get

$$\rho_p(\mathcal{A}|_{V_k}) \leqslant \lim_{r \to \infty} \max_{\substack{\mu \in \mathbb{N}_0^s,\, |\mu| = k+1 \\ 1 \leqslant j \leqslant n}} \left\| e_j^T \widetilde{\nabla}^\mu S_{\boldsymbol{A}} \boldsymbol{\delta} I_n \right\|_p^{1/r}.$$

The reverse inequality follows as in the scalar case. $\square$

The result of Lemma 4.8 is not satisfactory for our further analysis. To prove the analog of Proposition 4.6, we need to show that

$$\rho_p(\mathcal{A}|_{V_k}) = \lim_{r \to \infty} \max_{\substack{\mu \in \mathbb{N}_0^s,\, |\mu| = k+1 \\ 1 \leqslant j \leqslant n}} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} e_j \right\|_p^{1/r}, \quad 1 \leqslant p \leqslant \infty,$$

which is the result of Lemma 4.9. Its proof for $k = 0$ is given in [5].

Lemma 4.9. *Under the assumptions of Lemma* 4.8 *we have*

$$\lim_{r \to \infty} \max_{\substack{\mu \in \mathbb{N}_0^s,\, |\mu| = k+1 \\ 1 \leqslant j \leqslant n}} \left\| e_j^T \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n \right\|_p^{1/r} = \lim_{r \to \infty} \max_{\substack{\mu \in \mathbb{N}_0^s,\, |\mu| = k+1 \\ 1 \leqslant j \leqslant n}} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} e_j \right\|_p^{1/r}.$$

Proof. For $p = \infty$, the claim follows by (1). Indeed, observe that computing $\max_{1 \leqslant j \leqslant n} \|e_j^T \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n\|_\infty$ or the maximum of $\max_{1 \leqslant j \leqslant n} \|\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} e_j\|_\infty$ is equivalent to determining the maximal in modulus entry of the matrix sequence $\widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n$.

Let $1 \leqslant p < \infty$. Fix $\mu \in \mathbb{N}_0^s$, $|\mu| = k + 1$, and $r \in \mathbb{N}$. For $\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)} = \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n \in \ell_0^{n \times n}(\mathbb{Z}^s)$, denote its i-th row and j-th column sequences by

$$\left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,\cdot} \in \ell_0^{1 \times n}\left(\mathbb{Z}^s\right), \quad i = 1, \ldots, n,$$

and

$$\left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{\cdot,j} \in \ell_0^{n}\left(\mathbb{Z}^s\right), \quad j = 1, \ldots, n,$$

respectively. Then we have

$$\begin{aligned} \max_{1 \leqslant i \leqslant n} \left\| e_i^T \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} I_n \right\|_p^p &= \max_{1 \leqslant i \leqslant n} \left\| e_i^T \widetilde{\nabla}^\mu \boldsymbol{A}^{(r)} \right\|_p^p = \max_{1 \leqslant i \leqslant n} \left\| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,\cdot} \right\|_p^p \\ &= \max_{1 \leqslant i \leqslant n} \sum_{\alpha \in \mathbb{Z}^s} \left| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,\cdot}(\alpha) \right|_p^p \\ &= \max_{1 \leqslant i \leqslant n} \sum_{j=1}^n \sum_{\alpha \in \mathbb{Z}^s} \left| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,j}(\alpha) \right|^p \end{aligned} \tag{37}$$

and

$$\begin{aligned} \max_{1 \leqslant j \leqslant n} \left\| \widetilde{\nabla}^\mu S^r_{\boldsymbol{A}} \boldsymbol{\delta} e_j \right\|_p^p &= \max_{1 \leqslant j \leqslant n} \left\| \widetilde{\nabla}^\mu \boldsymbol{A}^{(r)} e_j \right\|_p^p = \max_{1 \leqslant i \leqslant n} \left\| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,\cdot} \right\|_p^p \\ &= \max_{1 \leqslant j \leqslant n} \sum_{\alpha \in \mathbb{Z}^s} \left| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{\cdot,j}(\alpha) \right|_p^p \\ &= \max_{1 \leqslant j \leqslant n} \sum_{i=1}^n \sum_{\alpha \in \mathbb{Z}^s} \left| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)_{i,j}(\alpha) \right|^p. \end{aligned} \tag{38}$$

Since $\boldsymbol{A}$ has compact support, the sum

$$\sum_{\alpha \in \mathbb{Z}^s} \left| \left(\widetilde{\nabla}^\mu \boldsymbol{A}^{(r)}\right)(\alpha) \right|^p$$

defines an $n \times n$ real matrix $C_{\mu,r}$ whose entries are sums of the absolute values of the corresponding elements of the matrices $(\widetilde{\nabla}^{\mu}\boldsymbol{A}^{(r)})(\alpha)$ raised to the p-th power. Note that the expressions in (37) and (38) are nothing else, but the ∞- and 1-norms of the matrix $C_{\mu,r}$, respectively. The claim follows due to the equivalence of matrix norms. $\square$

Now we are ready to prove the main result of this section. The proof of Proposition 4.10 for the case $k = 0$ is given in [5].

Proposition 4.10. *Under the assumptions of Lemma* 4.8 *and for* $\boldsymbol{B}_{k+1} \in \ell_0^{s^{k+1}n \times s^{k+1}n}(\mathbb{Z}^s)$ *satisfying* (5) *we get*

$$\rho_p(S_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}) = \rho_p(\mathcal{A}|_{V_k}).$$

Proof. Let $r \in \mathbb{N}$. The proof proceeds as in the scalar case. Here, we only highlight the differences between scalar and vector cases. Due to Lemmas 4.8 and 4.9, to prove the claim of this proposition we need to show that the quantities $\|S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}\|_p$ and $\max_{\substack{\mu\in\mathbb{N}_0^s,\,|\mu|=k+1\\ 1\leqslant j\leqslant n}} \|\widetilde{\nabla}^{\mu} S^r_{\boldsymbol{A}}\delta e_j\|_p$ are equivalent. Let $j = 1,\ldots,n$. As in the scalar case, by Proposition 4.1 and (5), we obtain

$$\begin{aligned}\big\|S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}\big\|_\infty &\geqslant C_1^{-1} \max_{\alpha\in 2^{r+1}K}\big|\nabla^{k+1}S^r_{\boldsymbol{A}}(\delta e_j)(\alpha)\big|_\infty \\ &= C_1^{-1} \max_{\substack{\mu\in\mathbb{N}_0^s,\,|\mu|=k+1\\ 1\leqslant j\leqslant n}} \big\|\widetilde{\nabla}^{\mu} S^r_{\boldsymbol{A}}\delta e_j\big\|_\infty\end{aligned}$$

with C_1 in (33). The last equality is due to $\operatorname{supp}(\nabla^{k+1}S^r_{\boldsymbol{A}}\delta I_n) \subset 2^{r+1}K$ with $K = [-2, N]^s$ and (34).

Let next $\boldsymbol{c} \in \ell^n_\infty(-2K)$ be the maximizing sequence in Proposition 4.1. Since the support of $\boldsymbol{c}$ is finite and $\|\nabla^{k+1}\boldsymbol{c}|_{-2K}\|_\infty = 1$, there exists a constant $C_2 > 0$ such that, by (30), we have

$$\big\|S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}\big\|_\infty \leqslant C_2 \max_{\alpha\in 2^{r+1}K}\left|\left(\sum_{\beta\in -2K}\sum_{j=1}^{n} S^r_{\boldsymbol{B}_{k+1}}\nabla^{k+1}(\delta e_j)(\cdot-\beta)\right)(\alpha)\right|_\infty.$$

As the sums above are finite and $\operatorname{supp}(S^r_{\boldsymbol{B}_{k+1}}\nabla^{k+1}\boldsymbol{\delta} I_n) \subset 2^{r+1}K$, we get by (34)

$$\big\|S^r_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}\big\|_\infty \leqslant C\cdot \max_{\substack{\mu\in\mathbb{N}_0^s,\,|\mu|=k+1\\ 1\leqslant j\leqslant n}} \big\|\widetilde{\nabla}^{\mu} S^r_{\boldsymbol{A}}\delta e_j\big\|_\infty, \quad C := C_2\cdot n\cdot|2K|.$$

For $1 \leqslant p < \infty$, modifications in the proof above are as in the scalar case. $\square$

4.5. Regularity analysis

In this subsection we briefly address the issue of characterizing the convergence and regularity of subdivision schemes using the restricted contractivity properties of the associated difference schemes. The main results of this and Sections 4.3 and 4.4 bring forth once again the fact that the smoothness of subdivision surfaces corresponds directly to the smoothness of the solutions of refinement equations. In general, this statement is true only under the assumption that the sum rules of certain order are satisfied by the corresponding mask, otherwise see [24, Remark 2.1].

In the scalar case, Proposition 4.6 and [24, Theorem 2.1] allow us to apply the result of [6, Theorem 4.1] and obtain the following.

Theorem 4.11. *Let* $1 \leqslant p \leqslant \infty$, $0 \leqslant k \leqslant N$, $\boldsymbol{A} \in \ell_0(\mathbb{Z}^s)$ *with* $\operatorname{supp}\boldsymbol{A} \subset [0, N]^s$. *Then* $S_{\boldsymbol{A}}$ *is* W^k_p*-convergent if and only if the mask* $\boldsymbol{B}_{k+1} \in \ell_0^{s^{k+1}\times s^{k+1}}(\mathbb{Z}^s)$ *as in* (5) *satisfies*

$$\rho_p(S_{\boldsymbol{B}_{k+1}}|_{\nabla^k}) < 2^{-k+s/p} \quad \big(\rho_\infty(S_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}) < 2^{-k} \textit{ for } p = \infty\big).$$

In the vector case, the assumption that the multiplicity m of the eigenvalue 1 of the mask symbol at zero is $1 \leqslant m \leqslant n$ does not allow for a straightforward use of the result of [6, Theorem 4.1]. Its generalization is stated in the following theorem.

Theorem 4.12. *Let* $1 \leqslant p \leqslant \infty$, $\boldsymbol{A} \in \ell_0^{n\times n}(\mathbb{Z}^s)$ *with* $\operatorname{supp}(\boldsymbol{A}) \subset [0, N]^s$ *satisfy sum rules of order* $k+1$ *and multiplicity* m, $0 \leqslant k < N$ *and* $1 \leqslant m \leqslant n$. *Then* $S_{\boldsymbol{A}}$ *is* W^k_p*-convergent if and only if the mask* $\boldsymbol{B}_{k+1} \in \ell_0^{ns^{k+1}\times ns^{k+1}}(\mathbb{Z}^s)$ *as in* (5) *satisfies*

$$\rho_p(S_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}) < 2^{-k+s/p} \quad \big(\rho_\infty(S_{\boldsymbol{B}_{k+1}}|_{\nabla^{k+1}}) < 2^{-k} \textit{ for } p = \infty\big).$$

The proof of Theorem 4.12 goes along the same lines as the one in [5] in the case $k = 0$. We do not present it here as it is out of scope of this paper. For details see [3].

4.6. Estimating JSR

In this subsection we present a method for estimating the restricted spectral radius $\rho_p(S_{\boldsymbol{B}_k}|_{\nabla^k})$ of the difference schemes $S_{\boldsymbol{B}_k}$, $k \geqslant 1$. In the case $p=\infty$, this method is an alternative to the one arising from the JSR approach, see Section 5. The comparison of methods in the case $1 \leqslant p < \infty$ is under further investigation. The goal of this subsection is solemnly in showing that the RSR approach does not only yield yet another characterization of subdivision convergence, but can also be of practical interest.

Note that by standard properties of spectral radii, it is less than $2^{-k+s/p}$ if there exists some $R \in \mathbb{N}$ such that $\|S^R_{\boldsymbol{B}_k}|_{\nabla^k}\|_p < 2^{R(-k+s/p)}$.

Consider the case $p=\infty$. Define the iterated mask $\boldsymbol{B}_k^{(r)}$ similarly to (3). Proposition 4.1 and the definition of $S_{\boldsymbol{B}_k}$ imply then for fixed $\alpha \in [0, 2^r-1]^s$

$$\left|S^r_{\boldsymbol{B}_k}\nabla^k \boldsymbol{c}(\alpha)\right|_\infty = \max_{i=1,\dots,ns^k}\left|\sum_{\beta\in[-N-k,1]}\bigl(B_k^{(r)}\bigr)_i\bigl(\alpha-2^r\beta\bigr)\nabla^k\boldsymbol{c}(\beta)\right|.$$

The structure of the mask $\boldsymbol{B}_k$ and ∇^k imply that, for $\beta \in [-N-k,1]^s$, we get

$$\begin{aligned}\bigl(B_k^{(r)}\bigr)_i\bigl(\alpha-2^r\beta\bigr)\nabla^k\boldsymbol{c}(\beta) &= \sum_{j=1}^{m}\sum_{\ell=1}^{s^k}\bigl(B_k^{(r)}\bigr)_{i,(\ell-1)n+j}\bigl(\alpha-2^r\beta\bigr)\widetilde{\nabla}^{\mathcal{M}_k(\ell)}\boldsymbol{c}_j(\beta)\\ &\quad+\sum_{j=m+1}^{n}\left(\sum_{\ell=1}^{s^k}\bigl(B_k^{(r)}\bigr)_{i,(\ell-1)n+j}\bigl(\alpha-2^r\beta\bigr)\right)\boldsymbol{c}_j(\beta).\end{aligned}$$

Thus, the restricted norm can be computed easily by solving the linear programming: solve for $\alpha \in [0,2^r-1]^s$ and over the rows $i=1,\dots,ns^k$ of $\boldsymbol{B}_k^{(r)}$ for $j=1,\dots,m$

$$\begin{aligned}&\max\sum_{\beta\in[-N-k,1]^s}\sum_{\ell=1}^{s^k}\bigl(B_k^{(r)}\bigr)_{i,(\ell-1)n+j}\bigl(\alpha-2^r\beta\bigr)\widetilde{\nabla}^{\mathcal{M}_k(\ell)}\boldsymbol{c}_j(\beta),\\ &\quad -1\leqslant\widetilde{\nabla}^{\mathcal{M}_k(\ell)}\boldsymbol{c}_j(\beta)\leqslant 1,\ 1\leqslant\ell\leqslant s^k,\ \beta\in[-N-k,1]^s\end{aligned}\tag{39}$$

and for $j=m+1,\dots,n$

$$\max\sum_{\beta\in[-N-k,1]^s}\left(\sum_{\ell=1}^{s^k}\bigl(B_k^{(r)}\bigr)_{i,(\ell-1)n+j}\bigl(\alpha-2^r\beta\bigr)\right)c_j(\beta),\quad -1\leqslant c_j(\beta)\leqslant 1.\tag{40}$$

It is easy to see that the maximum for $j=m+1,\dots,n$ is attained when

$$c_j(\beta) := \operatorname{sgn}\sum_{\ell=1}^{s^k}\bigl(B_k^{(r)}\bigr)_{i,(\ell-1)n+j}\bigl(\alpha-2^r\beta\bigr),\quad \beta\in[-N-k,1]^s.$$

For $j=1,\dots,m$ the above problem can be solved by standard methods of linear optimization, as there are only finitely many parameters and side conditions involved. The individual optimal solutions for $1\leqslant j\leqslant n$ are then added together and the values depending on i and α are maximized with respect to $i=1,\dots,ns^k$ and $\alpha\in[0,2^r-1]^s$.

Let us compare the above algorithm with the computation of the non-restricted norm

$$\bigl\|S^r_{\boldsymbol{B}_k}\bigr\|_\infty := \max_{\substack{\|\boldsymbol{d}\|_\infty=1\\ \boldsymbol{d}\in\ell_\infty^{ns^k}(\mathbb{Z}^s)}}\bigl\|S^r_{\boldsymbol{B}_k}\boldsymbol{d}\bigr\|_\infty.\tag{41}$$

The maximization problem in (41) can be reformulated as linear optimization problems for $i=1,\dots,ns^k$ and $\alpha\in[0,2^r-1]^s$

$$\begin{aligned}&\max\sum_{\beta\in[-N-k,1]^s}\sum_{j=1}^{ns^k}\bigl(B_k^{(r)}\bigr)_{ij}\bigl(\alpha-2^r\beta\bigr)d_j(\beta)\\ &\quad -1\leqslant d_j(\beta)\leqslant 1,\ \beta\in[-N-k,1]^s,\ j=1,\dots,ns^k.\end{aligned}\tag{42}$$

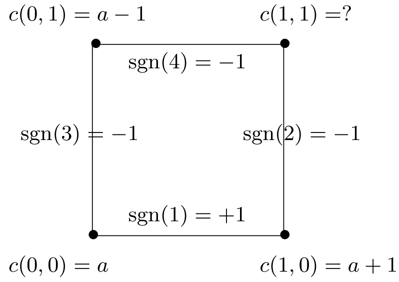

Fig. 1. Restricted vs. non-restricted norm, $a \in \mathbb{R}$.

The solution $\boldsymbol{d} \in \ell_\infty^{ns^k}([-N-k,1]^s)$ of this optimization problem is given by $d_j(\beta) = \mathrm{sgn}(B_k^{(r)})_{ij}(\alpha - 2^r\beta)$, $\beta \in [-N-k,1]^s$. In other words, the non-restricted ∞-norm can be computed as follows

$$\|S^r_{\boldsymbol{B}_k}\|_\infty = \max_{\alpha\in[0,2^r-1]^s}\left\{\left|\sum_{\beta\in\mathbb{Z}^s}\left|B_k^{(r)}(\alpha+2\beta)\right|\right|_\infty\right\}, \tag{43}$$

see [13]. Certainly, computing $\|S^r_{\boldsymbol{B}_k}\|_\infty$ is more efficient than solving (39)–(40) and we wish to replace (39)–(40) by (43) whenever possible. Unfortunately in general we have

$$\left\|S^r_{\boldsymbol{B}_k}|_{\nabla^k}\right\|_p \leqslant \left\|S^r_{\boldsymbol{B}_k}\right\|_p, \quad 1 \leqslant p \leqslant \infty.$$

In the case $p=\infty$, the equality is achieved only if there exists a sequence $\boldsymbol{c} \in \ell_\infty^n(\mathbb{Z}^s)$ satisfying (39) and

$$\widetilde{\nabla}^{\mathcal{M}_k(\ell)}\boldsymbol{c}_j(\beta) = \mathrm{sgn}\big(B_k^{(r)}\big)_{i,(\ell-1)n+j}\big(\alpha - 2^r\beta\big).$$

The following simple example shows that it is not always possible in the multivariate case. Let $s=2$, $i=k=n=r=1$ and $\boldsymbol{B} := \boldsymbol{B}_1$. If we assume the pattern of the signs of the entries of the mask $\boldsymbol{B}$ as in Fig. 1

(1) $\mathrm{sgn}\, B_{1,1}\big(\alpha - 2(1,0)\big) = +1$,
(2) $\mathrm{sgn}\, B_{1,2}\big(\alpha - 2(1,1)\big) = -1$,
(3) $\mathrm{sgn}\, B_{1,2}\big(\alpha - 2(0,1)\big) = -1$,
(4) $\mathrm{sgn}\, B_{1,1}\big(\alpha - 2(1,1)\big) = -1$,

then we see that the sequence $\boldsymbol{c} \in \ell_\infty^n(\mathbb{Z}^s)$ satisfying (39) and (40) does not exist. The effort of computing the restricted norm can be reduced in the following case.

Remark 4.13. We have $\|S^r_{\boldsymbol{B}_k}|_{\nabla^k}\|_\infty = \|S^r_{\boldsymbol{B}_k}\|_\infty$, if for all i, α and j the entries $B^{(r)}_{k,i,(\ell-1)n+j}(\alpha - 2^r\beta)$, $\beta \in \mathbb{Z}^s$, $\ell = 1,\ldots,s^k$, are all of the same sign, say $+1$.

For $1 \leqslant p < \infty$, by Proposition 4.2 and similar to the case $p=\infty$, the estimation of the restricted p-norm translates into the nonlinear optimization problem

$$\max \sum_{\alpha\in 2^{r+1}K}\left|\sum_{\beta\in\widetilde{K}} B_k^{(r)}\big(\alpha - 2^r\beta\big)d(\beta)\right|_p^p$$

with side conditions

$$\sum_{j=1}^{s^k}\sum_{\beta\in\widetilde{K}}\left|d_j(\beta)\right|_p^p \leqslant 1, \quad \widetilde{\nabla}^{\mathcal{M}_k(j)}\boldsymbol{d}_\ell = \widetilde{\nabla}^{\mathcal{M}_k(\ell)}\boldsymbol{d}_j,\ 1 \leqslant j < \ell \leqslant s^k,$$

where it is convenient to write $\boldsymbol{d} \in \ell_p^{ns^k}(\mathbb{Z}^s)$ in the form $\boldsymbol{d} = [\boldsymbol{d}_1,\ldots,\boldsymbol{d}_{s^k}]^T$ with $\boldsymbol{d}_\ell \in \ell_p^n(\mathbb{Z}^s)$. This optimization problem lies within the scope of *concave minimization problems*, i.e. the function to be minimized is concave and it is minimized over a convex compact set E described by a finite number of linear and nonlinear *convex constraints* of the type $g_i(\cdot) \leqslant 0$, $1 \leqslant i \leqslant M$, $M \in \mathbb{N}$. Problems of this type can be solved, for example, by means of outer approximation methods as given in [18, Algorithm 3.3]. To apply this method, we must verify that the functions f_α, defined for $\alpha \in 2^{r+1}K$ as

$$f_\alpha\big(\big[d_j(\beta)\colon 1 \leqslant j \leqslant s^k,\ \beta\in\widetilde{K}\big]^T\big) = \left|\sum_{\beta\in\widetilde{K}} B_k^{(r)}\big(\alpha - 2^r\beta\big)d(\beta)\right|_p$$

are convex with respect to the parameters $d_j(\beta)$ which immediately follows by the triangular inequality. Since also the functions $g(x) = x^p$ are convex for $x \geqslant 0$, so are the composite functions $g \circ f_\alpha$. Hence, the target function of the optimization problem is convex, and we face a convex maximization problem that is equivalent to a concave minimization problem.

Remark 4.14. In practice the cases $p = \infty$ and $p = 2$ are of special interest. In the case $p = 2$, see [16,23] for more efficient methods for estimating the Sobolev regularity of subdivision schemes.

5. Examples

In this section we illustrate the results of Proposition 3.3 and Section 4.6 with two examples.

Example 5.1. We consider the C^1-convergent bivariate vector subdivision scheme introduced and studied in [8]. The non-zero part of the corresponding subdivision mask is given by

$$\boldsymbol{A} = \frac{1}{8}\begin{array}{cccc} & \begin{bmatrix}1&0\\0&1\end{bmatrix} & \begin{bmatrix}1&2\\0&2\end{bmatrix} & \begin{bmatrix}0&2\\0&1\end{bmatrix} \\ \begin{bmatrix}1&0\\1&0\end{bmatrix} & \begin{bmatrix}4&0\\2&2\end{bmatrix} & \begin{bmatrix}4&2\\1&4\end{bmatrix} & \begin{bmatrix}1&2\\0&2\end{bmatrix} \\ \begin{bmatrix}1&0\\2&0\end{bmatrix} & \begin{bmatrix}4&0\\4&1\end{bmatrix} & \begin{bmatrix}4&0\\2&2\end{bmatrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix} \\ \begin{bmatrix}\mathbf{0}&\mathbf{0}\\\mathbf{1}&\mathbf{0}\end{bmatrix} & \begin{bmatrix}1&0\\2&0\end{bmatrix} & \begin{bmatrix}1&0\\1&0\end{bmatrix} & \end{array}.$$

The mask $\boldsymbol{A}$, due to the different definitions of $S_{\boldsymbol{A}}$, is the transpose of the mask in [8]. The mask here is shifted so that its left bottom entry in bold corresponds to the index $(0,0)$. The results of [8] imply that the scheme satisfies the sum rules (4) of order 2 and multiplicity 1. Proposition 3.1 and (8) yield the corresponding polynomial eigensequences $\boldsymbol{x}_\mu := \boldsymbol{x}_{1,\mu} \in \ell^2(\mathbb{Z}^2)$, $|\mu| \leqslant 2$, of $S_{\boldsymbol{A}}$

$$\begin{aligned}
&x_{(0,0)}(\alpha) = [\,1 \quad 1\,]^T, \\
&x_{(1,0)}(\alpha) = [\,1 \quad 1\,]^T \alpha_1 + \left[\,\tfrac{3}{2} \quad 2\,\right]^T, \qquad x_{(0,1)}(\alpha) = [\,1 \quad 1\,]^T \alpha_2 + \left[\,\tfrac{3}{2} \quad 2\,\right]^T, \\
&x_{(2,0)}(\alpha) = [\,1 \quad 1\,]^T \alpha_1^2 + 2 \cdot \left[\,\tfrac{3}{2} \quad 2\,\right]^T \alpha_1 + \left[\,\tfrac{8}{3} \quad \tfrac{8}{3}\,\right]^T, \\
&x_{(1,1)}(\alpha) = [\,1 \quad 1\,]^T \alpha_1 \cdot \alpha_2 + \left[\,\tfrac{3}{2} \quad 2\,\right]^T \alpha_1 + \left[\,\tfrac{3}{2} \quad 2\,\right]^T \alpha_2 + \left[\,\tfrac{17}{6} \quad \tfrac{17}{6}\,\right]^T, \\
&x_{(0,2)}(\alpha) = [\,1 \quad 1\,]^T \alpha_2^2 + 2 \cdot \left[\,\tfrac{3}{2} \quad 2\,\right]^T \alpha_2 + \left[\,\tfrac{8}{3} \quad \tfrac{8}{3}\,\right]^T.
\end{aligned}$$

The scheme is C^1, therefore, we show how to compute the transformation $\widehat{T}$ applying the algorithm of Proposition 3.3 for $|\mu| \leqslant 1$ only. To do that we solve the systems (12) and determine the unknown coefficients of

$$\hat{c}(\xi) = \frac{1}{4}\left(c(0,0) + c(1,0)e^{-i\xi_1} + c(0,1)e^{-i\xi_2}\right)$$

to be $c(0,0) = 8$, $c(1,0) = -2$ and $c(0,1) = -2$. One checks easily that the scheme $S_{\widetilde{\boldsymbol{A}}}$ given by (13) with

$$\widehat{T}(\xi) := \begin{bmatrix} 1 & 0 \\ -\hat{c}(\xi) & 1 \end{bmatrix}, \quad \xi \in \mathbb{R},$$

has the polynomial eigensequences as in (8)–(9).

Next we show how to use the linear programming in (39)–(40) to compute the restricted norm of a difference operator.

Example 5.2. The mask

$$\boldsymbol{A} = \frac{1}{8}\begin{array}{cccc} & \begin{bmatrix}1&1\\0&0\end{bmatrix} & \begin{bmatrix}2&1\\0&1\end{bmatrix} & \begin{bmatrix}1&0\\0&1\end{bmatrix} \\ \begin{bmatrix}1&1\\0&0\end{bmatrix} & \begin{bmatrix}4&2\\0&2\end{bmatrix} & \begin{bmatrix}5&1\\0&3\end{bmatrix} & \begin{bmatrix}2&0\\0&1\end{bmatrix} \\ \begin{bmatrix}2&1\\0&1\end{bmatrix} & \begin{bmatrix}5&1\\0&3\end{bmatrix} & \begin{bmatrix}4&0\\0&2\end{bmatrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix} \\ \begin{bmatrix}\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{1}\end{bmatrix} & \begin{bmatrix}2&0\\0&1\end{bmatrix} & \begin{bmatrix}1&0\\0&0\end{bmatrix} & \end{array}$$

is derived from the mask in [9] by means of the similarity transformation $\widehat{T}(\xi) = \left[\begin{smallmatrix}1 & 0\\ 1 & -1\end{smallmatrix}\right]$. The resulting eigensequences of $S_{\boldsymbol{A}}$ satisfy (8)–(9). For the case $k = 1$ see [3]. One of the possible second difference masks $\boldsymbol{B} := \boldsymbol{B}_2$ satisfying (5) for $k = 2$ is given by its symbol $\boldsymbol{B}^*(z) = [b_{i,j}(z)]_{1 \leqslant i,j \leqslant 8}$ whose non-zero entries are

$$\begin{aligned}
&b_{1,1}(z) = -2 + 2z_2 + z_1z_2 + 4z_2^2 + 2z_1z_2^2 + z_1z_2^3, \qquad b_{12}(z) = (1-z_1)^2 a_{12}(z),\\
&b_{1,3}(z) = 3 - 3z_1^2, \qquad b_{2,2}(z) = b_{4,4}(z) = b_{6,6}(z) = b_{8,8}(z) = a_{22}(z),\\
&b_{3,3}(z) = b_{5,5}(z) = 1 + z_2 + z_1 + 2z_1z_2 + z_1z_2^2 + z_1^2z_2 + z_1^2z_2^2,\\
&b_{3,4}(z) = b_{5,6}(z) = (1-z_1)(1-z_2)a_{12}(z),\\
&b_{7,7}(z) = 1 + 2z_1 + z_1^2 + z_1z_2 + 2z_1^2z_2 + z_2z_1^3, \qquad b_{7,8}(z) = (1-z_2)^2 a_{12}(z).
\end{aligned}$$

We have

$$\|S_{\boldsymbol{B}}\|_\infty = \left|B^{(2)}_{(0,0)}\right|_\infty = 2. \tag{44}$$

By Remark 4.13 we get

$$\|S_{\boldsymbol{B}}|_{\nabla^2}\|_\infty = \|S_{\boldsymbol{C}}\|_\infty = \frac{2}{3} > \frac{1}{2},$$

where the symbol $C^*(z) = [c_{i,j}(z)]_{1\leqslant i,j\leqslant 8}$ of the mask $\boldsymbol{C}$ differs from that of $\boldsymbol{B}$ only in the following entries

$$c_{1,1}(z) = 1 + 2z_2 + z_1z_2 + z_2^2 + 2z_1z_2^2 + z_1z_2^3 \quad \text{and} \quad c_{1,3}(z) = 0.$$

Thus, we cannot yet conclude that the scheme $S_{\boldsymbol{A}}$ is W^1_∞-convergent. We consider next the iterated mask $\boldsymbol{B}^{(2)}$ defined as in (3) with $\boldsymbol{B}^{(1)} := \boldsymbol{B}$. We get

$$\left\|S^2_{\boldsymbol{B}}\right\|_\infty = \left|B^{(2)}_{(0,2)}\right|_\infty = \frac{86}{64} > \frac{1}{2}. \tag{45}$$

To demonstrate our optimization approach, let us compute the restricted norm of $S^2_{\boldsymbol{B}}$ next and compare it with the value of the non-restricted norm we obtained in (45). Note that the value $\|S^2_{\boldsymbol{B}}\|_\infty$ in (45) is determined by the first row of $|B^{(2)}_{(0,2)}|_\infty$, analogously for

$$\left|B^{(2)}_{\varepsilon}\right|_\infty > \frac{1}{2} \quad \text{with } \varepsilon = (0,0), (2,0), (0,1), (1,2), (0,3).$$

This implies that it suffices to check if the restricted norm is less than $\frac{1}{2}$ for the above values of ε and $i = 1$. We only present the details for the case $\varepsilon = (0,2)$. The restricted norms for the other values of ε are computed similarly. Using (39)–(40) for $\varepsilon = (0,2)$, $i = 1$ and $j = 1$, we maximize the following function

$$\begin{aligned}
&B^{(2)}_{1,1}(0,2)\widetilde{\nabla}^{(2,0)}c_1(0,0) + B^{(2)}_{1,3}(0,2)\widetilde{\nabla}^{(1,1)}c_1(0,0) + B^{(2)}_{1,3}(4,2)\widetilde{\nabla}^{(1,1)}c_1(-1,0)\\
&\quad + B^{(2)}_{1,1}(0,6)\widetilde{\nabla}^{(2,0)}c_1(0,-1)
\end{aligned}$$

over the unknowns $x := [c_1(\beta)]^T_{\beta\in[-2,0]^2}$. Therefore, we need to solve the following linear programming

$$\max_x \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{64} & \frac{2}{64} & -\frac{1}{64} & -\frac{3}{64} & \frac{6}{64} & -\frac{3}{64} \end{bmatrix} x =: \max_x f^T x,$$

with the side conditions

$$Cx := \begin{bmatrix} C_1 \\ -C_1 \end{bmatrix} x \leqslant \mathbf{1} =: b, \quad C_1 = \begin{bmatrix}
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1\\
1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1\\
1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1
\end{bmatrix},$$

where $\mathbf{1}$ is a 20×1 column vector of ones. To solve this problem we use MATLAB routine $x = linprog(f, C, b)$. We get that $f^T x = \frac{4}{64}$. Thus, combining this with the value obtained for $j = 2$ and the other values of ε we get $\|S^2_{\boldsymbol{B}}|_{\nabla^2}\|_\infty = \frac{1}{64}(f^T x + 12) = \frac{1}{4} < \frac{1}{2}$. Therefore, by Theorem 4.12, the scheme $S_{\boldsymbol{A}}$ is W^1_∞-convergent.

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